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# A Hopf algebraic approach to TKK Lie algebras

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# Dutch preface

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# Introduction

A good conception of what structurable algebras and (Jordan-)Kantor pairs should be over commutative unital rings $\Phi$, with $1/6$ not necessarily contained in $\Phi$, is still lacking. This is a pity, as these structures play an important role in the construction of 5-graded Lie algebras and related algebraic structures. So, the investigations of this thesis should be considered in the context of a search of an extended definition for those structures, without any assumptions on the base ring $\Phi$ over which we work. Throughout the rest of this introduction, $\Phi$ will be the base ring.

We develop some novel concepts which allow us to generalize the results of certain articles authored by Faulkner [Fau00] and [Fau04]. In those articles, Faulkner establishes a connection between (quadratic) Jordan pairs, Hopf algebras and algebraic $\Phi$-groups. Through the generalization of these results, it becomes clearer what the (Jordan-)Kantor pairs should be if $1/6 \notin \Phi$. Moreover, if we can assume that $1/2 \in \Phi$, then it is highly probable that the Jordan-Kantor-like sequence pairs form an adequate generalization. If $1/2 \notin \Phi$, it is probable that some additional constraints are required.

Now, we give an outline of the structure and the results of our thesis. This outline will not be in exact order since most chapters do not really build upon chapters other than chapters 2 and 4. There is an occasional reference between the other chapters, but that is most of the time nothing of substance. Specifically, one can get to the main result(s) of each of those chapters without needing the others.

In chapter 2 we introduce some structures called sequence ($\Phi$-)groups and sequence pairs. These will be the core concepts that we use throughout this thesis. To set these up, especially the sequence pairs, we need to do some preliminary investigations which encompass a lot of that chapter. The sequence pairs form a proper generalization of the Jordan-Kantor pairs if $1/30 \in \Phi$.

**Theorem A.** *Let $P$ be a Jordan-Kantor pair ($1/6 \in \Phi$), and let $L$ be TKK($P$, $\textbf{\textit{InStr}}(P) + \Phi\zeta$) with $\zeta$ a grading element. Consider the $\Phi$-groups $G_+, G_-$ formed by the exponentials of only positively or only negatively graded elements. If either*

- $1/5 \in \Phi$,
- $x_n[a, b] = \sum_{i+j=n}[x_i a, x_j b]$ *for all* $a, b \in L$ *and* $n \in \mathbb{N}$, $x \in G_\pm(\Phi)$,

*then these groups form a sequence pair. Conversely, if $1/6 \in \Phi$, then each sequence pair is isomorphic to a sequence pair defined from a Jordan-Kantor pair.*

*Proof.* This is Theorem (2.4.8) combined with Corollary (4.3.5). ■

In the same chapter, we also prove that all Hopf algebras of a certain class induce sequence pairs.

**Theorem B.** *Let $H$ be a $\mathbb{Z}$-graded Hopf algebra over $\Phi$ so that the primitive elements have an induced 5-grading. Suppose that for all $\pm 2$ graded primitive elements $x$ there exist an infinite homogeneous dps over $x$ and that, either*

- $1/2 \in \Phi$ *and for each primitive element which is $\pm 1$ graded, there exists an infinite positive, or negative, homogeneous dps* $(1, x, \ldots)$,

- *there exists a quadratic form $f$ such that for all primitive elements $x$ which are $\pm 1$ graded there exists an infinite positive homogeneous dps $(1, x, f(x), \ldots)$,*

*then the positive homogeneous divided power series and the negative homogeneous divided power series form a Jordan-Kantor-like sequence pair. Conversely, for any sequence pair over a field $\Phi$ of characteristic different from $2$ and $3$, the universal sequence pair representation is such a Hopf algebra.*

*Proof.* The first direction is proved (modulo the trivial Jordan-Kantor-like addition) in Theorem (2.4.23). The converse part is proved by Corollaries (4.1.13) and (5.3.15). ■

We immediately integrated the converse parts of these theorems. To do that, we needed chapters 4 and 5. In the first section of chapter 4, we introduce the universal sequence pair representation, which is actually a Hopf algebra satisfying the conditions of Theorem B, excluding the restrictions on the primitive elements. In the same chapter, we investigate the (Jordan-)Kantor-like sequence pairs ($1/2 \in \Phi$ is assumed if it is Jordan-Kantor-like). This investigation shows, among other things, that each sequence pair is a Jordan-Kantor-like sequence pair if $1/6 \in \Phi$. We also investigate the class of structurable algebras from a hermitian form.

In chapter 5, we set ourselves up to prove Corollary (5.3.15), which we used in Theorem B. We prove the corollary by generalizing some work of Faulkner [Fau00, Section 6]. The last section of chapter 5 closely follows Faulkner. The first 2 sections serve to generate the tools to closely follow his exposition. In chapter 6, we generalize section 7 of the same article, by determining what the universal representation should be if $\Phi$ is a field of characteristic $0$.

**Theorem C.** *Let $G$ be a sequence pair over a field $\Phi$ of characteristic $0$. The universal sequence pair representation of $G$ is isomorphic to the universal enveloping algebra of the universal central extension of TKK$(G, \textbf{\textit{InDer}}(G))$.*

*Proof.* This is a slightly different formulation of Theorem (6.2.2). ■

We apply and generalize the results of another article of Faulkner [Fau04] in chapter 7. The structure of that article allows us to use a lot of the results without any adaptation. As such, the generalization fits into a single chapter. We finish that chapter by generalizing his last two theorems about Jordan pairs to Jordan-Kantor-like sequence pairs. We do not fully generalize that article, even though we could. As we do not fully generalize that article, this chapter remains independent of chapter 5. We now formulate the two main theorems of that chapter.

**Theorem D (7.5.1).** *If $G = (G_+, G_-)$ is a finite dimensional Jordan-Kantor-like sequence pair over $\Phi$, $J$ is the kernel of the TKK representation, and*

$$I = \ker(\epsilon) \cap J \cap S(J),$$

*then $G' = G_{U(G),I}$ is an algebraic $\Phi$-group, with algebraic $\Phi$-subgroups*

$$U^+ = G_{\mathcal{X},I^+} \cong G_+, \quad U^- = G_{\mathcal{Y},I^-} \cong G_-, \quad H = G_{\mathcal{H},I^0},$$

*with $I^+ = \mathcal{X} \cap I$, etc.*

**Theorem E (7.5.4).** *If $G$ is an affine algebraic group scheme, then every generalized elementary action of $\Phi_m$ on $G$ gives a $\mathbb{Z}$-grading of Dist$(G)$ as a Hopf algebra, such that the induced $\mathbb{Z}$-grading of Lie$(G)$ is*

$$\mathit{Lie}(G) = \mathit{Lie}(U^-)_2 \oplus \mathit{Lie}(U^-)_1 \oplus \mathit{Lie}(H) \oplus \mathit{Lie}(U^+)_1 \oplus \mathit{Lie}(U^+)_2$$

*and there is a homogeneous divided power sequence over each $x \in$ Lie$(U^{\pm})$. Moreover,*

$$(Lie(U^{+}), Lie(U^{-}))$$

*is a Jordan-Kantor-like sequence pair.*

There are 2 chapters we did not mention yet, namely chapters 3 and 8. These chapters do not generalize results from the articles we generalize in the other chapters. In chapter 3, we investigate special sequence pairs. This leads to some very palpable examples of sequence pairs. However, in that chapter we prove nothing out of the ordinary. We prove that associative algebras with involution certainly induce sequence pairs if $1/2 \in \Phi$. If $1/2 \notin \Phi$ we give some examples which include, for example, separable field extensions of degree 2 and quaternion algebras.

In chapter 8, we define derivations and determine the conditions derivations should satisfy. Furthermore, we revisit what Jordan-Kantor-like sequence pairs should be. This gives us a class of sequence pairs with which we can identify TKK Lie algebras so that they have defining representations in the endomorphism algebra.

**Theorem F (8.2.5).** *Let $G$ be a Jordan-Kantor-like sequence pair. For each derivation algebra $\mathcal{D}$ of $G$ containing the inner derivations, $L =$ TKK$(G, \mathcal{D})$ is a 5-graded Lie algebra and $G$ has a Jordan-Kantor-like sequence pair representation in the endomorphism algebra of $L$.*

To summarise, we have established back and forth correspondences (although not necessarily for all $\Phi$) between (certain classes of) (1) sequence pairs, (2) Hopf algebras, (3) Jordan-Kantor pairs, (4) Lie algebras and (5) algebraic $\Phi$-groups. Besides that, we also constructed some very concrete examples of sequence pairs.

# 1 Preliminaries and Context

In this chapter, we introduce some definitions and theorems. These definitions and theorems place the following chapters in context and introduce the notational conventions we used.

## 1.1 Conventions

We will always use $\Phi$ to denote a commutative unital ring. Additional assumptions, like $\Phi$ containing $1/6$, will always be stated clearly. Since fields are also commutative unital rings, we do not denote $\Phi$ differently if we are working over fields. We mean with $\Phi$-**alg** the category of unital commutative associative $\Phi$-algebras, and we will mostly denote its elements with $K$. Other $\Phi$-algebras do not need to be associative nor commutative. The reasons for these conventions are quite simple. We will be working with $\Phi$-algebras $A$. These $A$ will very often be either associative or Lie algebras. For such $A$, we will frequently be interested in the algebras $A \otimes_\Phi K$ seen as a $K$-algebra.

We assume that all $\Phi$-modules $M$ are unital, i.e. $1 \cdot m = m$. However, if we consider modules of $\Phi$-algebras $A$, then we do not necessarily make that assumption for the $A$-module structure. It would even be an impossible assumption, as $A$ does not necessarily contain a unit.

We mean with the **dual numbers** the ring $\Phi[\epsilon]$ with $\epsilon^2 = 0$. So, in the context of the dual numbers $\epsilon$ is always a well-determined element, except if we explicitly choose to use another description of the dual numbers.

For groups, we mean by the conjugation $g^h = h^{-1}gh$ and by the commutator $[g, h] = g^{-1}h^{-1}gh$.

## 1.2 Morphisms, substructures and gradings

We will need to introduce some notions of morphisms. To prevent stating the same thing ten times, we give a fairly general definition that applies to a lot of cases.

**Definition 1.2.1.** Consider a set $I$ and let $M$ and $N$ be $\Phi$-modules. Suppose that we have linear maps

$$f_i^X : X^{\otimes n_i} \longrightarrow X^{\otimes m_i},$$

for $X = N, M$ and $i \in I$. A **(homo-)morphism** between $(M, (f_i^M)_{i\in I})$ and $(N, (f_i^N)_{i\in I})$ is a linear map $\psi : M \longrightarrow N$ such that

$$f_i^N \circ \psi^{\otimes n_i} = \psi^{\otimes m_i} \circ f_i^M$$

holds for all i. The substructures of $M$ can be identified with the images of morphisms into $M$. The notions of monomorphism, epimorphisms and isomorphism can either be defined in the category-theoretic sense or by looking if the underlying linear map $\psi$ is a mono-, epi- or isomorphism.

*Remark* 1.2.2. Not all structures of this chapter fall under the previous definition. Nevertheless, it encompasses a lot of structures. For example, it covers Lie algebras, Lie triple systems, Hopf algebras, etc. To see that it encompasses these structures, we must identify bilinear multiplications $A \times A \longrightarrow A$ with linear maps $A \otimes A \longrightarrow A$, etc. A class of algebras that do not fall under the previous definition are the quadratic Jordan pairs, as they have a quadratic operation.

Now, we define graded operations.

**Definition 1.2.3.** Suppose that $M$ is a $\Phi$-module. Assume that there is an abelian group $G$ so that there are submodules $M_g$ for $g \in G$ of $M$ such that $M = \bigoplus_{g \in G} M_g$. Suppose that

$$f_i : M^{\otimes n_i} \longrightarrow M^{\otimes m_i}$$

are $\Phi$-linear maps for $i$ in some indexing set $I$. We call $(M, (f_i)_{i \in I})$ $G$-graded if

$$f_i(M_{a_1} \otimes \cdots \otimes M_{a_{n_i}}) \subset \bigoplus_{\mathbf{b}} M_{b_1} \otimes \cdots \otimes M_{b_{m_i}},$$

where $\mathbf{b} = (b_1, \dots, b_{m_i})$ runs over the solutions of $\sum_{i=1}^{m_j} b_j = \sum_{j=1}^{n_i} a_j$, is satisfied for all $i \in I$.

*Remark* 1.2.4.

- The $G$ of the previous definition will, throughout this thesis, always be $\mathbb{Z}$.
- This definition includes the bilinear multiplication of algebras. To see this, one needs to identify the multiplication with the corresponding linear map $M \otimes M \longrightarrow M$. We will denote this multiplication as $\mu : M \otimes M \longrightarrow M$ for associative $\Phi$-algebras $M$.
- Note that this definition takes into consideration multiple operations $f_i$. This will be useful when we consider Hopf algebras.

## 1.3 Lie algebras

**Definition 1.3.1.** A $\Phi$-module $L$ with a bilinear map $(x, y) \longmapsto [x, y] \in L$ is called a **Lie algebra**, if $[x, x] = 0$ and if it satisfies the **Jacobi identity:**

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0,$$

for all $x, y, z \in L$.

**Definition 1.3.2.** A **derivation** on a Lie algebra $L$ is a $\Phi$-module morphism $D : L \longrightarrow L$ such that $D[x, y] = [Dx, y] + [x, Dy]$. Derivations of the form $\mathrm{ad}(x)(y) = [x, y]$ are called **inner derivations**. Note that for each $x \in L$, the map $\mathrm{ad}(x)$ actually is a derivation.

**Definition 1.3.3.** For $x \in L$ we denote with $\exp(x) = \sum_{i=0}^{\infty} \mathrm{ad}(x)^i / i!$. Notice that this is not always well defined. However, if $\mathrm{ad}(x)^i = 0$ for all $i > j$ and if $j!$ is invertible in $\Phi$, then the exponentials are still defined.

*Remark* 1.3.4. We remark that any associative algebra $A$ induces a Lie algebra with Lie bracket $[a, b] = ab - ba$.

We will need universal central extensions. We will not really delve into the theory of those extensions. However, all results involving universal central extensions that we will use, come from Benkart and Smirnov [BS03]. So, we include the notions introduced in [BS03, Paragraph 5.9].

**Definition 1.3.5.** Let $L$ be a Lie algebra over $\Phi$. The **center** of $L$ is the submodule

$$Z(L) = \{x \in L | \text{ ad } x = 0\}$$

of $L$. We call $L$ **perfect** if $[L, L] = L$.

**Definition 1.3.6.** A **central extension** of a Lie algebra $L$ is a pair $(\tilde{L}, \pi)$ such that $\pi : \tilde{L} \longrightarrow L$ is a surjective morphism of Lie algebras with $\ker(\pi) \subseteq Z(\tilde{L})$. A **covering** is a central extension which is perfect. A cover is **universal** if for every central extension $(M, \tau)$, there is a unique homomorphism $\phi : \tilde{L} \longrightarrow M$ such that $\tau \circ \phi = \pi$. We refer to the universal central covering as the **universal central extension**.

*Remark* 1.3.7. If $L$ is a $\mathbb{Z}$-graded Lie algebra such that $L_{-m}, L_m$ are trivial for each $m > n$, then we also call $L$ a $(2n+1)$-graded Lie algebra.

## 1.4 Lie triple systems

**Definition 1.4.1.** Let $L$ be a $\Phi$-module together with a trilinear map $(x, y, z) \longmapsto [xyz]$. This is called a **Lie triple system** (LTS), if

$$0 = [xxz], \tag{LTS1}$$

$$0 = [xyz] + [yzx] + [zxy], \tag{LTS2}$$

$$[uv[xyz]] = [[uvx]yz] + [x[uvy]z] + [xy[uvz]], \tag{LTS3}$$

for all $u, v, x, y, z \in L$.

*Remark* 1.4.2. The axiom LTS3 might seem a bit odd at first glance. This axiom will, at least in the preliminaries, appear in multiple equivalent forms. We will see that it expresses that $L(u, v)(x) = [uvx]$ is a derivation.

**Definition 1.4.3.** A **derivation** of a Lie triple system $L$ is a $\Phi$-morphism $D : L \longrightarrow L$ such that

$$[D, L(x, y)] = L(Dx, y) + L(x, Dy),$$

with $L(x, y)(z) = [xyz]$. Set $\Theta(L)$ to be the derivation algebra of $L$ and let $\mathcal{G}$ be the submodule of $\Theta(L)$ generated by the derivations $L(x, y)$. All the $L(x, y)$ are derivations by axiom LTS3. Note that $\mathcal{G}$ is, by definition, an ideal of $\Theta(L)$.

*Construction* 1.4.4. Let $\mathcal{H}$ be a subalgebra of $\Theta(L)$ such that $\mathcal{G} \leq \mathcal{H}$. Consider

$$\mathcal{L}(\mathcal{H}, L) = \mathcal{H} \oplus L,$$

with product

$$[h_1 \oplus l_1, h_2 \oplus l_2] = ([h_1, h_2] + L(x_1, x_2)) \oplus (h_1 x_2 - h_2 x_1),$$

for $l_1, l_2 \in L, h_1, h_2 \in H$.

**Theorem 1.4.5 (Theorem VI.1 [Mey72]).** *For a Lie triple system $L$ and subalgebras $\mathcal{H}$ of $\Theta(L)$ such that $\mathcal{G} \leq \mathcal{H}$, the algebra $\mathcal{L}(\mathcal{H}, L)$ is a Lie algebra with involution $h \oplus l \mapsto -h \oplus l$. Moreover, $[xyz] = [[x, y], z]$ holds for all $x, y, z \in L$.*

**Definition 1.4.6.** The Lie algebra $\mathcal{L}(\mathcal{G}, L)$ is called the **standard embedding** of a LTS $L$.

*Remark* 1.4.7. We will use Lie triple systems for virtually all of the TKK constructions in the preliminaries. We will mostly use the standard TKK construction using the standard embedding. However, for different derivation algebras $\mathcal{H}$ satisfying the conditions of Theorem (1.4.5), we can also consider $\mathcal{L}(\mathcal{H}, L)$ and still call it the TKK construction.

## 1.5 Hopf algebras

Suppose that $A$ is a unital $\Phi$-algebra. We can think about the multiplication as a linear map

$$\mu : A \otimes A \longrightarrow A.$$

The unit can be thought of as a morphism

$$\eta : \Phi \longrightarrow A$$

given by

$$\eta(\lambda) = \lambda \cdot 1_A.$$

So, we can think of an algebra $A$ as a $\Phi$-module with certain maps $\mu, \eta$. Properties like associativity can be expressed as $\mu \circ (\mu \otimes \mathrm{Id}) = \mu \circ (\mathrm{Id} \otimes \mu)$.

Similarly, one defines a **coalgebra** $A$ from a **comultiplication**

$$\Delta : A \longrightarrow A \otimes A$$

and **counit**

$$\epsilon : A \longrightarrow \Phi.$$

The fact that $\epsilon$ is a counit, means that

$$(\mathrm{Id} \otimes \epsilon) \circ \Delta = \mathrm{Id} = (\epsilon \otimes \mathrm{Id}) \circ \Delta.$$

A coalgebra is **coassociative** if

$$(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta.$$

It is **cocommutative** if $\tau \circ \Delta = \Delta$ with $\tau(a \otimes b) = b \otimes a$.

**Definition 1.5.1.** Suppose that $A$ is both an associative algebra and a coassociative coalgebra with unit and counit such that $\Delta, \epsilon$ are algebra morphisms. In that case, we call $A$ a **bialgebra**.

An **antipode** $S : A \longrightarrow A^{op}$ on a bialgebra $A$ is an algebra morphism (or equivalently an algebra anti-morphism $S : A \longrightarrow A$) satisfying

$$\mu \circ (S \otimes \mathrm{Id}) \circ \Delta = \eta \circ \epsilon = \mu \circ (\mathrm{Id} \otimes S) \circ \Delta.$$

**Definition 1.5.2.** A **Hopf algebra** is a bialgebra with an antipode.

**Definition 1.5.3.** For a coalgebra $C$, a coideal is a linear subspace $I \subset C$ such that

$$\Delta(I) \subset I \otimes C + C \otimes C.$$

Note that for each coideal $I$ of $C$, $C/I$ also forms a coalgebra. A Hopf ideal of a Hopf algebra $H$ is an ideal $I$ which is, at the same time, a coideal and satisfies $S(I) \subset I, \epsilon(I) = 0$. We note that $H/I$ is a Hopf algebra too.

*Remark* 1.5.4. Observe that we did not require the Hopf algebras in consideration to be commutative. As such, we are considering a broader class than the Hopf algebras which are coordinate algebras of affine group schemes.

**Definition 1.5.5.** Let $H$ be a Hopf algebra and $x = (1, x_1, x_2, \ldots)$ a sequence of elements in $H$ such that $\Delta(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i}$. We call such an $x$ a **divided power series** or shortly a **dps**. If we write 'let $x$ be a dps', we mean an infinite dps with elements denoted as $x_i$. We call $a$ such that $(1, a)$ forms a divided power series, **primitive** elements. Elements $g$ such that $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ are called **group like**. We denote the submodule of primitive elements in $H$ as $\mathcal{P}(H)$.

## 1.6 Jordan algebras

We split the Jordan algebras into linear and quadratic Jordan algebras, even though these are equivalent structures if $1/2 \in \Phi$. Both structures are important. The linear Jordan algebras are a prime example of structurable algebras. The quadratic Jordan algebras display the link with (quadratic) Jordan pairs.

### 1.6.1 Linear Jordan algebras

We introduce some definitions and remarks from McCrimmon [McC06].

**Definition 1.6.1.** A **Jordan algebra** over $\Phi$, with $1/2 \in \Phi$ is a $\Phi$-algebra $J$ equipped with a commutative bilinear product, designated $xy$, which satisfies the **Jordan identity**:

$$[x^2, y, x] = 0,$$

where $[x, y, z] = (xy)z - x(yz)$ denotes the **associator**.

**Definition 1.6.2.** We say that equalities depending on elements of a certain $\Phi$-algebra $M$ hold **strictly**, if these equalities not only hold with elements in $M$ but hold also with general elements of $M \otimes K$, for all $K \in \Phi$-**alg**. With a **linearization** of a polynomial $p$ of homogeneous degree $n$ in $x$, we mean any term $p_i$ of

$$p(x + \lambda y) = p(x) + \lambda p_1(x, y) + \lambda^2 p_2(x, y) + \ldots + \lambda^n p(y),$$

or any linearization of such a $p_i$ seen as a homogeneous polynomial in $x$ or $y$. If $p$ is homogeneous of degree 2, then we often call $p_1(x, y)$ the **polarization** of $p$.

**Proposition 1.6.3.** *If $p = q$ is an identity between homogeneous polynomials of degree $n$ on $M$, then $p = q$ holds strictly if and only if all linearizations of this identity hold.*

*Proof.* Suppose $p = q$ holds strictly, then $p = q$ on $M \otimes K[t]$. In particular, $p(x + ty) = q(x + ty)$ holds for all $x, y \in M$. This means that

$$p(x) + tp_1 + \ldots + t^{n-1} p_{n-1} + t^n p(y) = q(x) + tq_1 + \ldots + t^{n-1} q_{n-1} + t^n q(y),$$

is satisfied strictly. We see that even $p_i = q_i$ must be satisfied strictly. So, all linearizations of those equations must also hold.

Suppose that the converse holds. We prove that the equations hold over $M \otimes K$. We know, for $x = x_1 \otimes k_1 + x_2 \otimes k_2$, that the equality is satisfied since

$$p(x) = p(x_1) \otimes k_1^n + p_1(x_1, x_2) \otimes k_1^{n-1} k_2 + \ldots + p(x_2) \otimes k_2^n,$$

by comparing terms belonging to $k_1^i k_2^j$. Similarly, we see that the equalities between linearizations of $p$ and $q$ still hold for $x$. We apply induction on the number of terms in $x$. Suppose now that $x = x_1 \otimes k_1 + y$ and that all linearizations of $p = q$ are satisfied for $y$. Then we can repeat the same process, to prove that $x$ satisfies the equations and all the equalities between all linearizations. So, we have shown that all $x \in M \otimes K$ satisfy the identities. Hence, the identities are satisfied strictly. ∎

*Remark* 1.6.4. It is possible to generalize the previous proposition to homogeneous maps, see Appendix B, which resolves the need for $M$ to be an algebra. This will allow us to use the equivalence between strictness for equations involving, for example, quadratic forms and the fact that certain equations involving the corresponding bilinear form must hold. Later, we will encounter another class of equations that can hold strictly.

*Remark* 1.6.5. We see that the Jordan identity is homogeneous of degree 3 in $x$ and linear in $y$. The fact that $1/2 \in \Phi$ means that the linearizations of the Jordan identity will hold. This means that any Jordan algebra satisfies the equations of the above definition strictly (cf. [McC06, Linearization Proposition II.1.8.5]).

*Example* 1.6.6. Let $A$ be an associative algebra, then $A$, with operation $(x, y) \mapsto \frac{xy+yx}{2}$, is a Jordan algebra (cf. [McC06, Full Example II.3.1.1]). We denote this algebra as $A^+$.

*Remark* 1.6.7. There are two important operators, namely $U_x = 2L_x^2 - L_{x^2}$, with $L_x(y) = xy$ the left multiplication, and $V_{x,y}z = U_{x,z}(y)$ with $U_{x,z}$ the polarization of $U_x$. These will exactly be the operators which make the linear Jordan algebra into a quadratic Jordan algebra. An explicit expression for $V_{x,y}$ is $2(L_{xy} - [L_x, L_y])$.

**Definition 1.6.8.** A Jordan algebra is **special** if it is a subalgebra of a Jordan algebra $A^+$ associated with an associative algebra $A$.

*Construction* 1.6.9. The **Tits-Kantor-Koecher (TKK) construction** for a linear Jordan algebra $J$, almost exactly in the form of [Mey70a, Satz 2.1] and [Mey70b], is taking the standard embedding, or any other embedding of Theorem (1.4.5), of the LTS $J \oplus \bar{J}$ with operation $[(a, b)(c, d)(e, f)] = (V_{a,d}e - V_{b,c}e, V_{b,c}f - V_{a,d}f)$. This construction gets its name from work of Jacques Tits[Tit62], Max Koecher [Koe67], and Isai Kantor [Kan64].

*Remark* 1.6.10. The description of the previous construction is right away fairly general. Meyberg [Mey70b] investigates the properties for this construction for **linear Jordan triple systems**. In [Mey70a] he continues that investigation for these triple systems, but also for **'verbundene paare'**, which are in some sense the linear Jordan pairs (although if 3 is a zero-divisor, it might be wise to add another axiom). We will leave out the definitions of those structures, as they are not terribly relevant.

The construction is entirely the same for their quadratic variants. Loos [Loo75, Introduction] refers [Loo79] (at the time a forthcoming paper) to establish how quadratic Jordan pairs relate to the Lie algebras from TKK construction, by linking them to certain group sheaves. However, there are easier ways to construct the corresponding Lie algebra, which do not establish a link with group sheaves. We prove that this construction works in the section on quadratic Jordan pairs.

### 1.6.2 Quadratic Jordan algebras

The quadratic Jordan algebras were initially introduced by McCrimmon [McC66].

**Definition 1.6.11.** A **unital quadratic Jordan algebra** is a $\Phi$-module $\chi$, with a quadratic map $U : \chi \longrightarrow \mathrm{End}_\Phi(\chi)$, with polarization $U_{x,z}(y) = \{x, y, z\} = V_{x,y}z$, and $1 \in \chi$, such that

- $U_1 = \mathrm{Id}$,
- $U_x U_y U_x = U_{U_x y}$,
- $U_x V_{y,x} = V_{x,y} U_x$,

for all $x, y \in \chi$ and such that these equalities remain valid under extension of scalars, i.e. are satisfied strictly. A (not necessarily unital) **quadratic Jordan algebra** is a submodule, closed under the operation

$$(x, y) \longmapsto U_x(y)$$

of a unital Jordan algebra $\chi$ together with an operation $x \mapsto x^2$ such that $U_x 1 = x^2$.

**Definition 1.6.12.** A homomorphism of a quadratic Jordan algebra is a linear map $\psi : J \longrightarrow J'$ such that

$$U_{\psi(x)}(\psi(y)) = \psi(U_x(y)).$$

This implies, using the polarization of $U$, that

$$\{\psi(x), \psi(y), \psi(z)\} = \psi\{x, y, z\},$$

for all $x, y, z$.

*Remark* 1.6.13. It is possible to axiomatize the (non-unital) quadratic Jordan algebras directly. A direct axiomatization and a proof that the definitions are equivalent are given in [Mey72, Theorem IX.1].

*Example* 1.6.14. Suppose $A$ is an associative algebra. Then $U_x y = xyx$ makes $A$ into a Jordan algebra with $x^2$ coinciding with the usual squaring operation on $A$. We call the quadratic Jordan algebras which are subalgebras of such Jordan algebras **special**.

*Remark* 1.6.15. If $1/2 \in \Phi$ the categories of linear Jordan algebras over $\Phi$ and quadratic Jordan algebras over $\Phi$ are equivalent. See, for example [Jac69, Section 1.4] or the computations preceding [Mey72, Section IX, Note in paragraph 9.5].

*Construction* 1.6.16. The TKK construction is the same as the TKK construction for Linear Jordan algebras (1.6.9). To determine the derivation algebras which may be used for the TKK construction, see Construction (1.7.4). The possibilities are restricted since the link with group sheaves introduces another notion of derivation, which confines the possible derivations.

## 1.7 Jordan pairs

We mostly follow Loos [Loo75]. Let $V^+, V^-$ be $\Phi$-modules and let

$$Q^\sigma : V^\sigma \longrightarrow \mathrm{Hom}_\Phi(V^{-\sigma}, V^\sigma),$$

be quadratic maps. We also consider maps

$$D^\sigma : V^\sigma \times V^{-\sigma} \longrightarrow \mathrm{Hom}_\Phi(V^\sigma, V^\sigma),$$

such that $D^\sigma_{x,y}(z) = Q^\sigma_{x,z}(y)$, with $Q^\sigma_{x,z}$ the polarization of $Q^\sigma$. Note that

$$D^\sigma_{x,y}(z) = D^\sigma_{z,y}(x) \quad \text{and} \quad D^\sigma_{x,y}(x) = 2Q^\sigma_x(y).$$

**Definition 1.7.1.** Let $V = (V^+, V^-)$ and $Q^\sigma$ be as just introduced, then $V$ is a **Jordan pair** if the following identities hold in all scalar extensions $V \otimes K$ of $V$:

1. $D^\sigma_{x,y} Q^\sigma_x = Q^\sigma_x D^{-\sigma}_{y,x}$,
2. $D^\sigma_{Q^\sigma_x(y),y} = D^\sigma_{x,Q^{-\sigma}_y(x)}$,

3. $Q^\sigma_{Q^\sigma_x(y)} = Q^\sigma_x Q^{-\sigma}_y Q^\sigma_x$.

In what comes, we will not write the signs $\pm\sigma$ any more, as the signs are uniquely determined once you fix one as $\sigma$. We also will denote $D_{x,y}(z) = \{x, y, z\}$. The **morphisms** are the linear maps satisfying the same condition as for quadratic Jordan algebras if we use the convention to not write the signs any more.

*Remark* 1.7.2. We know that a variant of axiom LTS3 of Lie triple systems must hold, specifically

$$\{u, v, \{x, y, z\}\} - \{x, y, \{u, v, z\}\} = \{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\} \tag{1.1}$$

must hold. If $1/6 \in \Phi$, then this equation is even equivalent with the axioms for Jordan pairs. If $1/2 \in \Phi$, we know that the first and second axiom, imply the third one. For a proof of these statements, see [Loo75, Proposition 2.1].

**Theorem 1.7.3.** *The TKK construction, applied to a quadratic Jordan pair, yields a Lie algebra.*

*Proof.* We consider $V^+ \oplus V^-$ together with $[xyz] = \{x, y, z\}$ if $x, z \in V^\sigma$ and $y \in V^{-\sigma}$. We set $[xyz] = -\{y, x, z\}$ if $y, z \in V^\sigma$ and $x \in V^{-\sigma}$. If $x, y \in V^\sigma$, we set $[xy\cdot] = 0$. Equation (1.1) shows that this satisfies axiom LTS3 of Lie triple systems. Axiom LTS1 is satisfied trivially. Axiom LTS2 also holds, since there are only 2 terms which are nonzero, and they are, necessarily, the same term but with opposite signs. So, we have an LTS and can use the standard embedding. ∎

*Construction* 1.7.4. We can not only use the standard embedding of the Lie triple system but all subalgebras, containing the inner derivations, of the algebra of all pairs of linear maps $\Delta = (\Delta_+, \Delta_-)$ such that

$$\Delta_\sigma Q^\sigma_x - Q^\sigma_x \Delta_{-\sigma} = V^\sigma_{\Delta_\sigma(x),x}.$$

We call this algebra the **derivation algebra**. All inner derivations $(V_{x,y}, -V_{y,x})$ satisfy the previous condition. This algebra might not be the full algebra $\Theta(L)$ of Theorem (1.4.5). This restriction corresponds exactly to the condition that $1 + \epsilon\Delta$ is an automorphism of the Jordan pair over the dual numbers. A reason for this choice is the fact that the TKK Lie algebra with those derivations, corresponds exactly to the Lie algebra of the algebraic group corresponding to the canonical Jordan system, corresponding to the Jordan pair (cf. [Loo79, Paragraph 5.14]) (if the modules $V^\pm$ are finitely generated projective modules). If you were to allow more derivations of the Lie triple system than the ones contained in the derivation algebra, the connection with group sheaves would not be so strong.

Now, we introduce some notions which are analogous to some new concepts we will introduce. These notions play a relatively important role in what we are trying to generalize.

**Definition 1.7.5.** For $(x, y) \in V^\sigma \times V^{-\sigma}$ we define the **Bergman operator** as

$$B(x, y) = \mathrm{Id} - D_{x,y} + Q_x Q_y.$$

**Definition 1.7.6.** We call $(x, y)$ quasi-invertible[1] if $B(x, y)$ is invertible.

**Proposition 1.7.7 (Proposition 3.2 [Loo75]).** *For $(x, y) \in V^\sigma \times V^{-\sigma}$, the following are equivalent:*

1. *$(x, y)$ is quasi-invertible.*
2. *There exists $z$ such that $B(x, y)z = x - Q_x y$ and $B(x, y)Q_z y = Q_x y$.*

[1] This is not the conventional definition (cf. [Loo75, Definition 3.1]). However, to use the conventional definition, we should introduce homotopes of Jordan pairs, which is something we will not need for anything else.

3. $B(x, y)$ *is invertible.*

4. $B(x, y)$ *is surjective.*

5. $2x - Q_x y$ *belongs to the image of* $B(x, y)$.

*If this is the case, then*

$$z = x^y = B(x, y)^{-1}(x - Q_x y),$$

*is the quasi-inverse of* $(x, y)$.

*Remark* 1.7.8. If $(x, y)$ is quasi-invertible, then $(y, x)$ is quasi-invertible with $y^x = y + Q_y x^y$ (cfr. [Loo75, Symmetry principle]).

Consider, for a Jordan pair $V$, the TKK Lie algebra $L$ together with a grading element. Then we can identify $x \in V^\sigma$ with the automorphism

$$\exp_\sigma(x) = \mathrm{Id} + \mathrm{ad}\, x + Q_x,$$

of $L$. Similarly we can identify automorphisms $h = (h_+, h_-)$ ov $V$, with automorphisms of $L$ by identifying it with $\tilde{h} = h_- + h_0 + h_+$, with $h_0 \cdot d = h^{-1} dh$, for 0-graded $d$. Notice that this action is well defined since an element of the 0-graded part of $L$ is fully determined by its action on the $\pm 1$-graded parts.

**Theorem 1.7.9 (Theorem 1.4 in [Loo95]).** *Let $V$ be a Jordan pair, $(x, y) \in V^\sigma \times V^{-\sigma}$, then $(x, y)$ is quasi-invertible if and only if there exists $(z, w) \in V^\sigma \times V^{-\sigma}$ and $h \in Aut(V)$ with*

$$\exp_+(x) \exp_-(y) = \exp_-(w) \tilde{h} \exp_+(z).$$

*In this case $z = x^y, w = y^x$.*

## 1.8 Associative pairs and special Jordan pairs

We introduce some concepts from Loos [Loo95, Paragraph 2.2].

**Definition 1.8.1.** An **associative pair** $S$ over $\Phi$ is a pair $S = (S^+, S^-)$ of $\Phi$-modules together with trilinear maps

$$S^\sigma \times S^{-\sigma} \times S^\sigma \longrightarrow S^\sigma : (x, y, z) \longmapsto xyz,$$

for $\sigma = \pm$, such that the associativity conditions

$$uv(xyz) = u(vxy)z = (uvx)yz$$

hold for all $u, x, z \in S^\pm$, $v, y \in S^\mp$.

*Remark* 1.8.2.

- Each associative pair can be embedded in an associative algebra.
- Each associative pair forms a Jordan pair under $Q_x(y) = xyx$.

**Definition 1.8.3.** A Jordan pair is called **special** if it is isomorphic to a Jordan subpair of an associative pair.

*Remark* 1.8.4.

- Equivalently, we could describe a special Jordan pair as a pair $(M^+, M^-)$ of submodules of an associative algebra $A$ closed under the operations

  $$M^\sigma \times M^{-\sigma} \longrightarrow M^\sigma : (x, y) \longmapsto xyx,$$

  for $\sigma = \pm$.
- The notion of specialness is a generalization of the notion of specialness of a quadratic Jordan algebra, which is, in itself, a generalization of the same notion for linear Jordan algebras.

## 1.9 Kantor pairs

For a triple system $T$, we mean by a **sign-grading** of $T$ a $\mathbb{Z}$-grading so that only the $\pm 1$-graded components are non-trivial. We could define Kantor pairs $(P^+, P^-)$ as sign-graded Lie triple systems $(P^+, P^-, [\cdot,\cdot,\cdot])$ (cfr. [AF99, Theorem 7]) where we forget everything, except the operations

$$V^+ : P^+ \times P^- \times P^+ \longrightarrow P^+,$$

and

$$V^- : P^- \times P^+ \times P^- \longrightarrow P^-,$$

both coinciding with $[\cdot,\cdot,\cdot]$. This would stress, immediately, the connection with TKK Lie algebras (namely take the standard embedding of the LTS). Alternatively, we could define it as pairs of modules with these operators, satisfying the axioms

$$[V^\sigma_{x,y}, V^\sigma_{u,v}] = V^\sigma_{V^\sigma_{x,y}u,v} - V^\sigma_{u,V^{-\sigma}_{y,x}v},$$

and

$$K_{a,b}V_{x,y} + V_{y,x}K_{a,b} = K_{K_{a,b}x,y}$$

with $K_{a,b}c = V_{a,c}b - V_{b,c}a$. It is customary to define them only over rings with $1/6$. However, Allison and Faulkner [AF99] define them over rings with $1/2$.

## 1.10 Structurable algebras

**Definition 1.10.1.** Suppose that $A$ is an algebra over $\Phi$. A linear map $x \mapsto \bar{x}$ is an involution on $A$ if it satisfies $\bar{\bar{x}} = x$ and $\overline{xy} = \bar{y}\bar{x}$ for all $x$ and $y$ in $A$.

Allison [All78] defined structurable algebras as a generalization of linear Jordan algebras. Let $\Phi$ be a field of characteristic different from 2 and 3 and $A$ a unital algebra over $\Phi$ with involution $a \mapsto \bar{a}$. Define

$$V_{x,y}z = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y.$$

We call the algebra $A$ **structurable** if

$$[V_{x,y}, V_{u,v}] = V_{V_{x,y}u,v} - V_{u,V_{y,x}v}.$$

*Remark* 1.10.2. We will not really consider morphisms of structurable algebras, but it is worth noting that we consider the involution as an integral part of the structure.

*Remark* 1.10.3. Suppose that the involution of a structurable algebra is trivial, then, as Remark (1.7.2) indicates, $(A, A)$ forms a Jordan pair[2], i.e. it is a Jordan triple system. The Jordan triple systems with a unit (or squaring operation) are exactly the Jordan algebras (cf. [Mey72, Theorem X.1][3]). Therefore, we see that the structurable algebras with trivial involution are exactly the linear Jordan algebras with a unit.

[2] In this remark we switch freely between quadratic and linear structures, as this is possible for the characteristics in consideration.

[3] This theorem says something about homotopes. The 1-homotope has the same operations as the Jordan triple system, and is a unital Jordan algebra for 1 with $U_1 = \mathrm{Id}$.

We mention the classification of the central simple structurable algebras. We will not define all of these classes, as the first three classes are the only ones that appear in this thesis.

*Example* 1.10.4. There are six classes of central simple structurable algebras:

1. Associative unital algebras $A$ with involution,
2. Linear Jordan algebras,
3. Hermitian structurable algebras (or structurable hermitian algebra of the hermitian form $h$ as [All79, Section 7] calls it), we introduce these in section 4.4,
4. Forms of tensor product of two composition algebras,
5. Structurable algebras of skew dimension 1,
6. Smirnov algebras.

We will not use the last three classes, so we will not introduce them.

*Remark* 1.10.5. Originally, Allison [All79, Theorem 11] proved that every central simple structurable algebra is one of the first 5 classes of Example (1.10.4), missing the sixth class, for fields of characteristic 0. Smirnov [Smi90a] classified these algebras for fields of characteristic different of 5, thereby noting [Smi90b] that Allison missed the Smirnov algebra. Building upon the work of Boelaert, De Medts and herself [BDMS19], Stavrova [Sta20], formulates a different classification which also includes characteristic 5. This different classification could lead to an extension of the classifications of [All79], [Smi90a] to include characteristic 5.

*Construction* 1.10.6. We could explicitly reformulate the TKK construction for structurable algebras, as Allison [All79, Section 3] did. However, it is far more convenient, in the context of this thesis, to consider the structurable algebras as a subset of the Kantor triple systems, i.e. $P$ with operation $\{\cdot,\cdot,\cdot\}$ such that $(P, P)$ with 2 times the same operation forms a Kantor pair (this can be seen as a consequence of [All79, Theorem 3] and the definition of a Kantor pair as a sign graded Lie triple system). This does not mean that there are no advantages in using the explicit construction of Allison. We will just not need those advantages in this thesis.

Suppose $L$ is a Lie algebra and $e, f, h$ are elements of $L$ such that

$$[h, f] = -2f, \quad [h, e] = 2e, \quad [e, f] = h,$$

then we call $(h, e, f)$ an $S$-triple in $L$. Note that the subalgebra generated by those three elements is a three dimensional subalgebra. For this Lie algebra there exists a standard family of finite dimensional modules, and they are the only ones for algebraically closed fields of characteristic 0 (cfr. [Hum72, Section II.7]). Consider the free modules $V^k$, with bases

$$\{v_{-k}, v_{-k+2}, \ldots, v_{k-2}, v_k\},$$

and action

$$h \cdot v_i = i v_i, \quad e \cdot v_i = \left(\frac{k+i}{2} + 1\right) v_{i+1}, \quad f \cdot v_i = \left(\frac{k-i}{2} + 1\right) v_{i-1}.$$

We are only interested in these modules over fields with characteristic $\neq 2$. However, this would also work for general rings, since the construction ensures that $k \equiv i \mod 2$ so that $\frac{k \pm i}{2}$ is a well defined integer.

**Theorem 1.10.7.** *Suppose $\Phi$ is a field of characteristic different from $2, 3$ and $5$, and $L$ is a finite dimensional Lie algebra over $\Phi$. Then, there exists a structurable algebra with involution $(A, \bar{\cdot})$ and a Lie algebra $D$ such that $L$ is isomorphic to the TKK Lie algebra of $A$ with zero-graded part $D$, if and only if $L$ contains an $S$-triple such that the algebra $\mathcal{G}$ generated by the $S$-triple satisfies the following conditions:*

1. *$L$ is the direct sum of copies of $V^1, V^3$ and $V^5$ as a $\mathcal{G}$-module under the adjoint action*
2. *$\mathcal{G}$ does not centralize any non-trivial ideal of $L$.*

*Proof.* This is [All79, Theorem 4]. ∎

*Remark* 1.10.8. One can specify exactly what $D$ can be using the derivation and the inner structure algebra. We will not use this theorem explicitly. However, since we are working with Jordan-Kantor pairs it is useful to keep in mind that we could use this theorem to identify which Jordan-Kantor pairs are, in fact, structurable algebras.

*Remark* 1.10.9. Benkart and Smirnov [BS03, Proposition 2.4], refer to the result of this theorem and its proof, without assuming that the characteristic of $\Phi$ is different from 5.

*Remark* 1.10.10. It is interesting to note, cf. [AF99, Corollary 15], that the Kantor pairs which admit, in some sense, a unit are exactly the Kantor pairs coming from structurable algebras. To make this correspondence work, one needs to generalize structurable algebras so that they are also defined over commutative associative unital rings containing $1/6$, as Kantor pairs are defined over such rings. This is done by Allison and Faulkner [AF93], by defining $A$ to be structurable if

$$[s, b, c] + [b, s, c] = 0$$

for $s = a - \bar{a}$ and $a, b, c \in A$, holds. This restriction is satisfied for fields of characteristic different from 2 and 3, cf. [All78, Proposition 1].

The same, namely that having a unit of some sort implies that it comes from an algebra, is also true for Jordan pairs and Quadratic Jordan algebras, cf. [Loo75, Proposition 1.11].

*Remark* 1.10.11. Despite not delving deeper into the theory for structurable algebras and focussing more on (Jordan-)Kantor pairs, the generalization of the structurable algebras from Allison and Faulkner [AF93] corresponds exactly to the structurable algebras which we investigate as Kantor pairs. Namely, [AF93, Theorem 4.1, Theorem 5.5] shows us that these are the algebras for which the TKK construction still works.

## 1.11 Jordan-Kantor pairs

We assume that $1/6 \in \Phi$. Jordan-Kantor pairs were introduced by Benkart and Smirnov [BS03]. Intuitively, they axiomatize what the non-zero graded parts of a 5-graded Lie algebra can be.

We consider a linear Jordan pair $J = (J^+, J^-)$. This is a quadratic Jordan pair where you forget the operators $Q$ and keep the operator $D$. The only axiom needed is (1.1) and that $D_{x,y}z = D_{z,y}x$. For $J$ we consider a special $J$-bimodule $M = (M^+, M^-)$ relative to an action $\cdot$. We mean by this that

$$j_\sigma \cdot m_{-\sigma} \in M_\sigma,$$

for $j_\sigma \in J^\sigma$ and $m_{-\sigma} \in M_{-\sigma}$, and that

$$D_{a,b}c \cdot m = a \cdot (b \cdot (c \cdot m)) + c \cdot (b \cdot (a \cdot m)).$$

From now onwards we will not write those brackets any more, as the brackets can only be placed in one meaningful way. Alternatively, one can interpret what we are doing as defining $abc = a(bc)$. We suppose, additionally, that there are anti-commutative bilinear maps

$$k : M_\sigma \times M_\sigma \longrightarrow J_\sigma.$$

The last operators we need to introduce are operators $V$ on $M$ which make $M$ into a Kantor pair. However, given the additional axioms we will impose, it is enough to require that equation (1.1) holds, with $V_{x,y}z =: \{x, y, z\}$.

We write $P = (J, M)$ for the structure with all these operators. Such a $P$ is a **Jordan-Kantor pair** if the following identities are satisfied

1. $k(x, z) \cdot y = V_{x,y}z - V_{z,y}x$,
2. $k(x, z) \cdot b \cdot u = V_{z,b\cdot x}u - V_{x,b\cdot z}u$,
3. $k(b \cdot x, y) \cdot z = b \cdot V_{x,y}z + V_{y,x}(b \cdot z)$,
4. $D_{a,b}k(x, z) = k(a \cdot b \cdot x, z) + k(x, a \cdot b \cdot z)$,
5. $D_{a,k(y,w)}c = k(a \cdot w, c \cdot y) + k(c \cdot w, a \cdot y)$,
6. $k(k(z, u) \cdot y, x) = k(V_{x,y}z, u) + k(z, V_{x,y}u)$,

for $x, u, z \in M^\sigma, y, x \in M^{-\sigma}, a, c \in J^\sigma, b \in J^{-\sigma}$.

Now, we define the structure algebra for $P$. Set

$$\mathcal{E} = \mathrm{End}_\Phi(J^-) \oplus \mathrm{End}_\Phi(M^-) \oplus \mathrm{End}_\Phi(M^+) \oplus \mathrm{End}_\Phi(J^+).$$

The structure algebra $\mathbf{Str}(J, M)$ is the set of $T \in \mathcal{E}$ such that

$$T \cdot F_{x,y} = F_{Tx,y} + F_{x,Ty},$$

for $F = D, V$, and

$$T(a \cdot b) = Ta \cdot b + a \cdot Tb, \quad Tk(a, b) = k(Ta, b) + k(a, Tb),$$

where $a, b, x, y$ are chosen in such a way that they are always contained in the domain of the operators.

For $a \in J^+, b \in J^-$, there exists a corresponding $\delta(a, b) \in \mathbf{Str}(J, M)$. For $(x, y) \in M^+ \times M^-$ there exists a similar such element which we will denote $v(x, y)$. Specifically, we define $\delta_{a,b}c = D_{a,b}c$, $\delta_{a,b}d = -D_{b,a}d$, $\delta(a, b)x = a \cdot (b \cdot x)$ and $\delta(a, b) \cdot y = -b \cdot (a \cdot y)$, for $a, c \in J^\sigma, b, d \in J^{-\sigma}, x \in M^\sigma$ and $y \in M^{-\sigma}$. For $x \in M^\sigma, y \in M^\sigma$ one has a similar $v(x, y)$ which acts on $M$ as expected and on $J^\sigma$ by $v(x, y)a = k(a \cdot y, x)$. It acts similarly on $J^{-\sigma}$, but with an opposite sign. These elements span the inner structure algebra $\mathbf{InStr}(J, M)$.

There is a unique way to construct a 5-graded Lie algebra $L(J, M, \mathcal{D})$ out of these elements once you fix a Lie subalgebra $\mathcal{D}$ of $\mathbf{Str}(J, M)$ containing $\mathbf{InStr}(J, M)$, namely put

$$L(J, M, \mathcal{D}) = J^- \oplus M^- \oplus \mathcal{D} \oplus M^+ \oplus J^+,$$

with the graded parts exactly corresponding to how we wrote $L(J, M, \mathcal{D})$ down. The Lie brackets not, or only, involving $\mathcal{D}$ are entirely determined from the operators $k, \cdot$. Moreover, $\mathcal{D}$ acts as a Lie algebra on the rest of $L(J, M, \mathcal{D})$ while the Lie brackets with a result that should be zero-graded are entirely determined by the elements $\delta(a, b), v(a, b)$.

**Theorem 1.11.1 (Theorem 4.3 of [BS03]).** *The space $L(J, M, \mathcal{D})$ is a Lie algebra. Moreover, $\mathcal{D}$ acts, under the adjoint map, faithful on the rest of $L(J, M, \mathcal{D})$.*

One can also prove the converse.

**Theorem 1.11.2 (Theorem 4.5 of [BS03]).** *Suppose $L$ is a 5-graded Lie algebra*

$$L = \bigoplus_{i=-2}^{5} L_{L_i},$$

*then*

- *$(L_2, L_{-2})$ form a Jordan pair with $[[a, b], c] = D_{a,b}c$,*
- *$(L_1, L_{-1})$ forms a special $J$-bimodule under $a \cdot x = [a, x]$,*
- *the pair $(J, M)$ with $V_{x,y}z = [[x, y], z]$ and $k(a, b) = [a, b]$ forms a Jordan Kantor pair, denoted $P(L)$,*
- *if the adjoint action of $L_0$ on the rest of the Lie algebra is faithful, then $L$ is isomorphic to a Lie algebra $L(P(L), \mathcal{D})$. If, additionally, $L$ is generated by $L_{\pm 1}, L_{\pm 2}$, then $\mathcal{D} = \boldsymbol{InStr}(P(L))$.*

In order to not confuse ourselves, we shall use TKK$(P, \mathcal{D})$ to designate $L(P, \mathcal{D})$, so that it is obvious that we consider the TKK Lie algebra.

We formulate a crucial property that allows us to identify which Jordan-Kantor pairs are Kantor pairs. We mean by the Jordan-Kantor pair associated with a Kantor pair, the unique Jordan-Kantor pair we get from applying the previous theorem on any TKK Lie algebra associated with the Kantor pair.

**Proposition 1.11.3 (Proposition 7.5 of [BS03]).** *A Jordan-Kantor pair $P = (J, M)$ is isomorphic to the Jordan Kantor pair associated to some Kantor pair $M$ if and only if*

- *$J$ acts faithfully on the bimodule $M$,*
- *$J_\sigma = k(M_\sigma, M_\sigma)$ for $\sigma = \pm$.*

*Remark* 1.11.4. There exists another class of Jordan-Kantorpairs, namely the **J-ternary** algebras. They were introduced by Allison [All76]. These are precisely the Jordan-Kantor pairs for which $J$ has units. This means that $J$ is two times the same linear Jordan algebra. For these, there is an analogous theorem to Theorem (1.10.7) where the modules are not $V^1, V^3, V^5$ but $V^1, V^2, V^3$, corresponding to the fact that the elements of the $S$-triple $e, f, h$ are $\pm 2$ or 0 graded, instead of $\pm 1$ or 0.

## 1.12 Jordan Pairs and Hopf algebras

We recall some essential theorems from the article [Fau00] written by Faulkner. These should allow the reader to interpret what we will do as a generalization of that article.

**Definition 1.12.1.** A dps $(1, x_1, x_2, \ldots)$ in a $\mathbb{Z}$-graded Hopf algebra is **homogeneous** if there exists a $\sigma = \pm$ such that each $x_i$ is $\sigma i$-graded. We remark that it is conventional to let $\sigma$ be any $\sigma \in \mathbb{Z}$, but we will not consider any such homogeneous divided power series.

**Theorem 1.12.2 (Theorem 5 of [Fau00]).** *If $A$ is a $\mathbb{Z}$-graded Hopf algebra such that*

- $\mathcal{P}(A) = \mathcal{P}_{-1} \oplus \mathcal{P}_0 \oplus \mathcal{P}_1$,
- *there is a homogeneous divided power sequence* $(1, x_1, x_2, \ldots)$ *over all* $x \in \mathcal{P}_\sigma$, $\sigma = \pm 1$,

*then* $(\mathcal{P}_1, \mathcal{P}_{-1})$ *is a Jordan pair with* $Q_x(y) = x_2 y - xyx + yx_2$, *with* $x_2$ *the second element of* ***the unique*** *homogeneous divided power sequence over* $x$.

In that article he considers binomial divided power maps. To be specific, let $V$ be a $\Phi$-module and let $A$ by a unital associative algebra over $\Phi$. A sequence

$$\rho = (\rho_0, \rho_1, \ldots)$$

of maps $\rho_n : V \longrightarrow A$ is a sequence of **binomial divided power** maps provided $\rho_0(v) = 1$ and $\rho_n$ is a homogeneous map (cfr. Appendix B) of degree $n$ whose $(i,j)$-linearization is $(u,v) \longmapsto \rho_i(u)\rho_j(v)$. We will denote $\rho_i(v)$ as $v_i$. These maps are best characterized in terms of the following corollary.

**Corollary 1.12.3 (Corollary 7 of [Fau00]).** *If* $\rho$ *is a sequence of maps* $\rho_n : v \longrightarrow v_n$ *with* $v_0 = 1$, *then* $\rho$ *is a sequence of binomial power divided maps if and only if for each extension* $K$ *of* $\Phi$ *there is an extension* $\rho'_n : V \otimes K \longrightarrow A \otimes K$ *satisfying*

- $(\lambda v)_n = \lambda^n v_n$,
- $(v+u)_n = \sum_{i+j=n} v_i u_j$,

*for all* $u, v \in V \otimes K$.

For such binomial divided power maps, we can define

$$\mathrm{ad}_x^{(n)}(y) = \sum_{i+j=n} (-1)^j x_i y x_j.$$

The **divided power representations** of a Jordan pair $V = (V^+, V^-)$ are exactly the pairs $\rho = (\rho^+, \rho^-)$ of binomial divided power maps from $V^\sigma$ to an associative unital $\Phi$-algebra $A$ such that for all extensions $K$ of $\Phi$ and all $x \in V^\sigma \otimes K$, $y \in V^{-\sigma} \otimes K$,

$$\mathrm{ad}_x^{(k)}(y_l) = \begin{cases} 0 & k > 2l, \\ Q_x(y)_l & k = 2l. \end{cases}$$

It is possible to construct the universal divided power representation $U$ for a Jordan pair $V$.

**Theorem 1.12.4 (Theorem 15 in [Fau00]).** *The universal divided power representation* $\gamma$ *in* $U$ *of a Jordan pair* $V$ *is a* $\mathbb{Z}$*-graded cocommutative Hopf algebra and* $\gamma_1^\sigma$ *is injective.*

Faulkner also proves that this universal representation, at least is the $V^\sigma$ are free $\Phi$-modules, satisfies the conditions of Theorem (1.12.2), cf. [Fau00, Corollary 28]. To achieve that, he makes use of the fact that for all $x \in V^+, y \in V^-$ we can define

$$\sum h_{p,q} s^p t^q = h = \exp(v)\exp(-sx)\exp(-ty)\exp(u),$$

with $u$ the quasi-inverse of $(sx, ty)$ and $v$ the quasi-inverse of $(ty, sx)$. The crucial property is that each binomial divided power representation satisfies the **exponential property**, i.e.

$$h_{pq} = 0 \quad \text{if } p \neq q.$$

These are not the only interesting results of that article. However, these give the necessary background to understand what this thesis generalizes. Specifically, **sequence Φ-groups** generalize divided power binomial maps. They generalize these maps in such a way that $V$ does not necessarily have to be a Φ-module but can be a group with a certain kind of multiplication. **Sequence pairs** correspond to divided power representations of a Jordan pair in the sense that there is a pairing between sequence groups with 2 operators $T$ and $Q$, instead of a single operator $Q$. We use these generalizations to generalize all the important theorems of the article. We will also generalize most of the results of another article [Fau04] of Faulkner.

# 2 Sequence groups and pairs

In this chapter, we introduce the main concepts that will be used throughout this thesis. Firstly, we introduce sequence groups. These generalize the notion of sequences of binomial divided powers, as introduced by Faulkner [Fau00, Section 5]. Secondly, we formulate an essential theorem, namely Theorem (2.3.3). This theorem provides us with all the necessary ingredients to define sequence pairs. The sequence pairs themselves are a generalization of divided power representations of Jordan pairs. Once the sequence pairs are defined, we prove that the two main classes of interest, namely Jordan-Kantor pairs (under a certain condition) and a certain class of Hopf algebras, define sequence pairs.

## 2.1 Sequence groups

**Definition 2.1.1.** Let $A$ be an associative unital $\Phi$-algebra. Suppose that $D \subset A^{\mathbb{N}}$ is a set of infinite sequences in $A$ with $d_0 = 1$ for all $d \in D$. Assume that $D$ is closed under the following operations

1. $\lambda \cdot (1, x_1, \ldots, x_n, \ldots) = (1, \lambda x_1, \ldots, \lambda^n x_n, \ldots)$,
2. $(1, x_1, \ldots, x_n, \ldots) \times (1, y_1, \ldots, y_n, \ldots) = (1, x_1 + y_1, \ldots, \sum_{i+j=n} x_i y_j)$,

for all $\lambda \in \Phi$. If $D$, together with the operation '$\times$', forms a group with unit $(1, 0, 0, \ldots)$, then we call $D$ a **sequence group** in $A$. We will sometimes denote elements $x$ of $D$, for which there exists a natural number $n$ such that $x_m = 0$ for all $m > n$, as $(1, x_1, \ldots, x_n)$ and drop all the zeros.

*Remark* 2.1.2. Notice that we can identify a sequence group in $A$ with a subgroup of the units in $A[[t]]$. Specifically, we can map

$$(1, x_1, x_2, \ldots) \longmapsto 1 + tx_1 + t^2 x_2 + \cdots .$$

This is a group isomorphism. Note, moreover, that the scalar multiplication on the sequence group is given by the substitution of $\lambda t$ for $t$. It will often be more useful to think about these groups as groups of sequences. In the context of power sequences, we mean by $\exp(tx) = 1 + tx_1 + t^2 x_2 + \cdots$. In fact, we will use the notation $\exp$ a bit more often. If the sequences all have finite length $< n$, then it is perfectly fine to consider $\exp(x) = 1 + x_1 + x_2 + \cdots$. The second exponential is not necessarily injective, so we should use that exponential carefully.

**Lemma 2.1.3.** *Let $G$ be a sequence group and $\lambda \in \Phi$. The map*

$$(\lambda \cdot) : G \longrightarrow G : g \longmapsto \lambda \cdot g$$

*is a group automorphism of $G$.*

*Proof.* Suppose that $d = (d_0, d_1, \ldots)$, $e = (e_0, e_1, \ldots)$ are elements of $G$ and $de = ((de)_0, (de)_1, \ldots)$. We compute

$$\lambda^n (de)_n = \lambda^n \sum_{i+j=n} d_i e_j = \sum_{i+j=n} \lambda^i d_i \lambda^j e_j = \sum_{i+j=n} (\lambda \cdot d)_i (\lambda \cdot e)_j = ((\lambda \cdot d)(\lambda \cdot e))_n.$$

Hence, the map $g \longmapsto \lambda \cdot g$ is a group automorphism. ∎

The notion of a sequence group will be useful. However, we do not want that such a group only exists in a fixed algebra $A$. To escape a fixed algebra $A$, we identify exactly what parts of the structure need to be preserved. There are three important components of these kinds of groups. Firstly, there is the group structure. Secondly, we have a multiplication with scalars. And lastly, we have a sequence of subgroups $H^i$, which are the subgroups of $G$ consisting out of sequences $h = (1, h_1, \ldots, h_i, \ldots)$ such that $h_j = 0$ if $j \leq i$. We note that $[G, H^i] \subset H^{i+1}$ for each $i$ and that each of these $H^i$ is normal in $G$.

**Definition 2.1.4.** We call a sequence group $G$ such that $H^n = 0$ a sequence group of **class** $n$. A group homomorphism $\phi : G \longrightarrow G'$ between sequence groups which preserves the scalar multiplication is called a **sequence group representation**[1]. If $G'$ is a sequence group in a unital, associative algebra $A$, then we call this morphism a sequence group representation in $A$. If the representation also satisfies $\phi(H^i) \subset H'^i$ and $\phi^{-1}(H'^i \cap \mathrm{Im}(\phi)) \subset H^i$, then we call the representation **faithful**. We call it **essentially faithful**, if only $\phi^{-1}(H'^i \cap \mathrm{Im}(\phi)) \subset H^i$.

**Proposition 2.1.5.** *Suppose that $G$ is a sequence group. Let $\rho : G \longrightarrow A^{\mathbb{N}}$ be a map, then $\rho$ is a sequence group representation if and only if the following three properties hold*

- $\rho(\lambda \cdot g) = \lambda \cdot \rho(g)$,
- $\rho(gh) = \rho(g) \times \rho(h)$,
- $\rho(1) = (1)$.

*Moreover, if $G$ is only an abstract group with a scalar multiplication, $\rho$ is injective and the previous properties hold, then we can endow $G$ with a sequence group structure such that $\rho$ is a faithful representation.*

*Proof.* We note that these properties are necessary. So, we prove that they are sufficient. First, we prove that $\rho(G)$ forms a sequence group. We note that $\rho(G)$ is closed under the multiplication and the scalar multiplication of Definition (2.1.1) if the first two properties hold. The third property ensures that $\rho(G)$ has the right unit and that $\rho(G)$ is closed under inverses. So, $\rho(G)$ is a group. Note that $\rho : G \longrightarrow \rho(G)$ is a morphism of groups by the second property. By the first property, $\rho$ is compatible with scalar multiplication. So, $\rho$ is a morphism of sequence groups.

For the moreover part, we note that there is a unique way to endow $G$ with subgroups $H^i$ such that $\rho : G \longrightarrow \rho(G)$ is a faithful sequence group representation. ■

*Remark* 2.1.6. If we choose to describe a sequence group $G$ first as an abstract group with some sort of scalar multiplication and then apply Proposition (2.1.5) on $\rho : G \longrightarrow A^{\mathbb{N}}$ to prove that it a sequence group, then it is useful to think about $\rho$ as a **defining representation**.

We will use $\rho : G \longrightarrow A$ to denote a sequence group representation of a sequence group $G$ in an algebra $A$ corresponding to the sequence of maps $\rho_i : G \longrightarrow A$, i.e. $\rho(g) = (\rho_0(g), \rho_1(g), \ldots)$. We will also use $g_n$ to denote the element $\rho_n(g)$.

For the next example, we rely on a generalization of the Campbell-Baker-Hausdorff theorem which allows us to carry over the results of this theorem to more general rings under certain conditions.

[1]This naming convention might seem strange. However, we will almost always think of $G'$ as a sequence group in a unital associative algebra $A$.

**Theorem 2.1.7 (Campbell-Baker-Hausdorff, Theorem 9 in [AF99]).** *Let $\mathcal{F}_\Phi\{x,y\}$ denote the free associative algebra generated by $x$ and $y$, and let $I_k$ be the ideal spanned by all homogeneous elements of total degree $\geq k$. If $1/n! \in \Phi$, let $\bar{z} = z + I_{n+1} \in \mathcal{F}_\Phi\{x,y\}/I_{n+1}$, and let $\exp(\bar{z}) = \sum_{k=0}^{n} z^k/k!$. Then*

$$\exp(\bar{x})\exp(\bar{y}) = \exp(\bar{w}),$$

*where*

$$w = x + y + [x,y]/2 + [[x,y],y]/12 - [[x,y],x]/12 + \dots,$$

*is an element of the Lie subalgebra of $\mathcal{F}_\Phi\{x,y\}$ generated by $x$ and $y$.*

*Example* 2.1.8. Suppose $1/6 \in \Phi$. Let $P$ be a Jordan-Kantor pair over $\Phi$, with associated 5-graded Lie algebra $L = \mathrm{TKK}(P, \mathbf{InStr}(P) + \Phi\zeta)$ containing a grading element $\zeta$. Consider the groups $G_\sigma = L_\sigma \times L_{2\sigma}$ with group operation

$$(a,b)\cdot(c,d) = (a+c, b+d+[a,c]/2)$$

and scalar multiplication

$$\lambda \cdot (a,b) = (\lambda a, \lambda^2 b).$$

We define $H_\sigma^1$ as $L_{2\sigma}$ and want to prove that $G_\sigma$ is a sequence group of class 2.

We now construct a faithful sequence group representation of $G_\sigma$. The algebra $E = \mathrm{End}_\Phi(L)$ is a 9-graded associative algebra if we consider $\phi \in E$ to be an element of $E_k$ (the submodule of elements which have as grading $k$) if $\phi(L_j) \subset (L_{j+k})$ for all $j$. Consider the map

$$\begin{aligned}\rho_\sigma(a,b) = {} & \text{'}\exp(a+b)\text{'} = (1, \exp(a+b)_\sigma, \dots, \exp(a+b)_{4\sigma}) \\ & = (1, (\mathrm{ad}\ a), (\mathrm{ad}\ a)^2/2 + (\mathrm{ad}\ b), \dots, \sum_{i+2j=4} (\mathrm{ad}\ a)^i/(i!)(\mathrm{ad}\ b)^j/(j!)),\end{aligned}$$

where $\exp(a)_i$ denotes the component of $\exp(a)$ in $E_i$. It is obvious that the first and third property listed in Proposition (2.1.5) hold. Furthermore, the representation is injective since $(\mathrm{ad}\ a)(h) = i\sigma a$ where $h$ is the grading element and $a \in L_{\sigma i}$. The second property of Proposition (2.1.5) is a consequence of the Campbell-Baker-Hausdorff theorem, as stated earlier, with $n = 4$.

**Lemma 2.1.9.** *If $\rho : G \longrightarrow A$ is a sequence group representation, then*

$$\hat{\rho} : G \longrightarrow \mathit{End}_\Phi(A),$$

*with*

$$\hat{\rho}_n(g)(a) = ad_g^{(n)}(a) = \sum_{k+\ell=n} g_k a (g^{-1})_\ell,$$

*is a sequence group representation. Moreover, the following equality holds*

$$ad_g^{(n)}(ab) = \sum_{k+\ell=n} ad_g^{(k)}(a)\, ad_g^{(\ell)}(b).$$

*Proof.* We use Proposition (2.1.5) to prove the first part. The first property of that proposition is easily verified by

$$\lambda^n \mathrm{ad}_g^{(n)}(a) = \sum_{i+j=n} \lambda^i g_i a \lambda^j (g^{-1})_j = \sum (\lambda\cdot g)_i a ((\lambda \cdot g)^{-1})_j = \mathrm{ad}_{(\lambda\cdot g)}^{(n)}(a),$$

where we used, in the second equality, that $g \mapsto \lambda \cdot g$ is a group automorphism and that $\lambda^i g_i = (\lambda \cdot g)_i$ in $A$. The second property holds since

$$\begin{aligned}\sum_{i+j=n} \mathrm{ad}_g^{(i)} \mathrm{ad}_h^{(j)}(a) &= \sum_{i+j+k+\ell=n} g_i h_j a (h^{-1})_k (g^{-1})_\ell \\ &= \sum_{c+d=n} (gh)_c a ((gh)^{-1})_d \\ &= \mathrm{ad}_{gh}^{(n)}(a).\end{aligned}$$

The third property is obvious. Therefore, $\hat{\rho}$ is a sequence group representation. Now, we prove the second statement of the lemma by computing

$$\begin{aligned}\mathrm{ad}_g^{(n)}(ab) &= \sum_{i+j=n} g_i ab (g^{-1})_j \\ &= \sum_{p+q+r+s=n} g_p a (g^{-1})_q g_r b (g^{-1})_s \\ &= \sum_{i+j=n} \mathrm{ad}_g^{(i)}(a) \mathrm{ad}_g^{(j)}(b),\end{aligned}$$

where the second equality holds since

$$\sum_{r+q=m} (g^{-1})_q g_r = (g^{-1} g)_m = (1)_m = \begin{cases} 0 & m > 0 \\ 1 & m = 0 \end{cases}. \qquad \blacksquare$$

**Definition 2.1.10.** We call the representation $\hat{\rho}$ constructed from a representation $\rho : G \longrightarrow A$ as executed in the previous lemma, the **adjoint representation**.

## 2.2 Sequence $\Phi$-groups

In this section, we identify a specific subclass of sequence groups which will be the sequence groups of interest during the rest of this thesis. The goal is to identify specific sequence groups $G$ of class 2 in algebras $A$ such that for all $K \in \Phi$-**alg**, there exists a well-determined $G(K)$ that is a sequence group in $A \otimes K$. We assume in this section that all $K$ are elements of $\Phi$-**alg**, while $A$ is just an associative unital $\Phi$-algebra.

We suppose that $G$ has a $\Phi$-module structure so that it is isomorphic to $G/H^1 \times H^1$ with operation $(a, b)(c, d) = (a + c, b + d + \psi(a, c))$ for a bilinear form $\psi$. We also assume that the scalar multiplication on $G$ is related to the $\Phi$-module structure by

$$\lambda \cdot (a, b) = (\lambda a, \lambda^2 b).$$

If we use a scalar multiplication without any reference to which one it is, it is the scalar multiplication on the group. If we want to use the module multiplication, which is almost always on $H^1$, we will state clearly that we consider that multiplication and if it is on $H^1$, we will often denote it as $\cdot_H$. If $G$ is of the previous form, then we call $G$ a **potential sequence $\Phi$-group**[2].

[2] This is a definition of which we only make use in this section. The goal of this section is to find polynomial equations which guarantee that a potential sequence $\Phi$-group has a faithful sequence $\Phi$-group representation. The notion used outside of this section, will exactly be the one of a sequence $\Phi$-group.

**Definition 2.2.1.** We call a sequence group representation of a potential sequence $\Phi$-group **even** if $\rho(H^1)_{2n+1} = 0$ for all $n$.

*Remark* 2.2.2. We assume henceforth that **all sequence group representations** of potential sequence $\Phi$-groups, **are even**.

Note that $f(a) = (a,b)_2 - b_2 = (a,0)_2$ is well defined for all $(a,\cdot) \in G$. We prove that it is a quadratic form. First, we note that $f(\lambda a) = (\lambda a, 0)_2 = \lambda^2 (a,0)_2$. We also see that $F(a,c) = f(a+c) - f(a) - f(c) = (a+c, \psi(a,c))_2 - \psi(a,c)_2 - (a,0)_2 - (c,0)_2 = a_1 c_1 - \psi(a,c)_2$. We prove that $F$ is a symmetric bilinear form. It is clearly bilinear. Moreover, we see that $F(a,c) = a_1 c_1 - \psi(a,c)_2$, while we know that $[a_1, c_1] = [a,c]_2 = \psi(a,c)_2 - \psi(c,a)_2$. Hence $F(a,c)$ is symmetric. This means that $f$ is a quadratic form.

Suppose that there exists a representation $G(\Phi) \longrightarrow A$. We want to find conditions which are equivalent to the existence of sequence group representations of $G(K) = (G/H^1 \times H^1) \otimes K$ in $A \otimes K$. We call representations $\rho : G(\Phi) \longrightarrow A$ which extend to representations $\rho : G(K) \longrightarrow A \otimes K$ **sequence $\Phi$-group representations**. If $H^1 = 0$ we already know what sequence $\Phi$-group representations are.

**Lemma 2.2.3.** *Suppose $\rho$ is a sequence group representation of a potential sequence $\Phi$-group $G(\Phi)$ with $H^1 = 0$ in a $\Phi$-algebra $A$. Then $\rho : G(K) \longrightarrow A \otimes K$ is a sequence group representation for all $K$ if and only if for all $u, v \in G(\Phi)$ and $k, l \in \mathbb{N}$ the following equalities hold*

- $[v_k, u_l] = 0$,
- $v_k v_l = \binom{k+l}{k} v_{k+l}$.

*Proof.* This is a reformulation of [Fau00, Lemma 6 and Corollary 7]. ■

*Remark* 2.2.4. If we assume that all representations are even, then we can give $H^1$ another structure as a sequence group, nonisomorphic to the structure of $H^1$ seen as a sequence subgroup of $G$. Specifically, $H^1$ can be seen as the sequence group $h \to (1, h_1, h_2, \ldots)$ which is embedded in $G$ by mapping $h \to (1, 0, h_1, 0, h_2, \ldots)$. This embedding is not a morphism of sequence groups, as the scalar multiplication is not the same. The first scalar multiplication on the first realization of $H^1$ as a sequence group, is the scalar multiplication on $H^1$ as a $\Phi$-module coinciding with the $\Phi$-module structure on $G$. Seen as a subgroup of $G$, it has another scalar multiplication. They are related by $\lambda \cdot_G h = \lambda^2 \cdot_H h$. The fact that $H^1$ can be seen as a sequence group with $(H^1)^1 = 0$, implies that if $G$ is a sequence $\Phi$-group, then $H^1$ satisfies Lemma (2.2.3).

With the previous remark in mind, we seek for a generalization of Lemma (2.2.3) to potential sequence $\Phi$-groups. Now, we identify two necessary conditions which correspond to specific properties of sequence $\Phi$-group representations, that generalize the conditions of the previous lemma. Later, we will prove these conditions to be sufficient.

**Lemma 2.2.5.** *If $\rho : G \longrightarrow A$ is a sequence $\Phi$-group representation, then the following equalities must hold for all $x, y \in G(\Phi)$:*

$$x_j x_i = \sum_{a+2b=i+j} \binom{a}{i-b} x_a (x_1^2 - 2x_2)_{2b}, \tag{2.1}$$

$$[x_j, y_i] = \sum_{\substack{a+c=i \\ b+c=j \\ c \neq 0}} y_a x_b [x,y]_{2c}, \tag{2.2}$$

*where $(x_1^2 - 2x_2)$ represents a well-determined element of $H^1(\Phi)$. Moreover, if these equations hold for a potential sequence $\Phi$-group $G'$, then we know that $\mu x \cdot \lambda x = (\mu + \lambda)x \cdot (\lambda\mu x_1^2 - 2\lambda\mu x_2)$ and $x(\lambda y) = (\lambda y)x[x, \lambda y]$ as sequences in $A \otimes K$ for all $K \in \Phi$-**alg** and $x, y \in G'(\Phi)$.*

*Proof.* We compute $h$ in $x \cdot \lambda x = (1 + \lambda)x \cdot h$. We know that $(x \cdot \lambda x)_2 = x_2 + \lambda x_1^2 + \lambda^2 x_2$. On the other hand, we get $(1 + \lambda)^2 x_2 = x_2 + \lambda 2x_2 + \lambda^2 x_2$. So, the difference of these two is $\lambda(2x_2 - x_1^2)$. We also know that the first two coordinates are equal. Therefore, we need $h \in H(K)$ with $h_2 = -\lambda(2x_2 - x_1^2)$. This $h$ exists, since $x(-x) = (1, 0, 2x_2 - x_1^2, \ldots)$.

Now, we identify a necessary and sufficient condition so that $x \cdot \lambda x = (1+\lambda)x \cdot h$ holds for $\lambda \in K$. Namely, for all $i$ and $n$, the terms belonging to $\lambda^i$ in the $n$-th coordinate should always be equal to each other. The $n$-th coordinate of $(1 + \lambda)x \cdot h$ equals $\sum_{a+2b=n}(1 + \lambda)^a x_a h_{2b}$. So, comparing the terms belonging to $\lambda^i$, we get

$$x_j x_i = \sum_{a+2b=i+j} \binom{a}{i-b} x_a (2x_2 - x_1^2)_{2b}.$$

This also proves the necessity of equality (2.1).

Now we prove that equality (2.2) must hold. This equality corresponds to $x(\lambda y) = (\lambda y)x[x, \lambda y]$. This means that for all $i$ and $n$, the terms in $(x \cdot \lambda y)_n = (\lambda y \cdot x[x, \lambda y])_n$ belonging to $\lambda^i$ should be the same. As $[x, \lambda y]$ equals $\lambda \cdot_H [x, y]$ where $\cdot_H$ denotes scalar multiplication on $H^1$ instead of the scalar multiplication on $G$, we get that the term belonging to $\lambda^i$ in the right hand side equals

$$\sum_{\substack{a+c=i \\ a+b+2c=n}} y_a x_b [x, y]_{2c}.$$

This yields equality (2.2), where we need to bring the term with $c = 0$ to the left hand side. ■

Note that the previous lemma generalizes the properties of the preceding lemma, as

$$[x, y], (x_1^2 - 2x_2) \in H^1 = 0$$

implies that a lot of the terms will be zero.

Now, we prove that a sequence group representation of a potential sequence $\Phi$-group, satisfying the two equations of Lemma (2.2.5) is a sequence $\Phi$-group representation. As we have already mentioned, the part $\rho : H^1 \longrightarrow A$ is already compatible with scalar extension.

**Proposition 2.2.6.** *A potential sequence $\Phi$-group $G$ with sequence group representation*

$$\rho : G \longrightarrow A$$

*has a sequence $\Phi$-group representation $G \longrightarrow A$ extending $\rho$ if and only if equalities (2.1) and (2.2) hold.*

*Proof.* These equalities are necessary. We prove the sufficiency.

We first define $\rho : G(K) \longrightarrow A \otimes K$. We already know that

$$\rho : H(K) \longrightarrow A \otimes K$$

is well defined and satisfies all necessary properties if equalities (2.1) and (2.2) hold. Note, additionally, that each $h \in H(K)$ commutes with every $g_m$ for $g \in G(\Phi)$, as $(\sum \lambda_i h_i)_n g_m$ equals

$g_m(\sum \lambda_i h_i)_n$ because each $(h_i)_j$ for $h_i \in H^1(\Phi)$ commutes with every $g \in G(\Phi)$. So, we define $\rho : G(K) \longrightarrow A \otimes K$ inductively, under the assumption that $G/H^1$ is a free $\Phi$-module with well-ordered basis[3] $(g_i)_{i \in I}$. We consider the well-ordering to be on $I$. We have already determined $\rho(0, h)$ for $(0, h) \in H^1(K)$. We suppose now that we have determined $\rho(g, h)$ for $(g, h) \in V_i(K) \times H^1(K)$ with $V_i$ the submodule of $G/H^1$ spanned by the elements $(g_j)_{j<i}$. Suppose that we have already defined $\rho$ for all basis elements which precede $g' = g_i$ in such a basis, then for all $\mu \in K$ and $g \in V_i(K)$ we define[4]

$$\rho_n(g + \mu g', h + \mu\psi(g, g')) = \sum_{i+j+2k=n} \mu^j g_i g'_j h_{2k},$$

where $g'$ has as representation $(1, g'_1, f(g'_1), \ldots)$. Observe that this definition satisfies

$$\rho_n(g, a + b) = \sum_{i+2j=n} \rho_i(g, a) b_{2j} = \sum_{i+2j=n} b_{2j} \rho_i(g, a),$$

using the fact that $b_{2j}$ commutes with every $g_i$ for $g \in G(K)$. Note that

$$g'g'' = (1, g'_1 + g''_1, f(g'_1 + g''_1) + \psi(g'_1, g''_1)_2, \ldots),$$

for $g', g'' \in G/H^1(\Phi)$. Moreover, for these specific elements we compute that $2g_2 - g_1^2 = -\psi(g, g)$, since $0 = f(0) = f(g_1) + f(-g_1) - g_1^2 + \psi(g, g) = 2g_2 - g_1^2 + \psi(g, g)$. So, equation (2.1) shows that

$$((\lambda g', 0)(\mu g', 0))_n = (\lambda g' + \mu g', \lambda\mu\psi(g', g'))_n$$

for all $g' \in G(\Phi)$.

It follows from the definition of $\rho_n$ and equation (2.1) that

$$((g, h)(g', h'))_n = ((g, 0), (g', 0)(0, h + h'))_n = (g + g', h + h' + \psi(g, g'))_n$$

for all $g, g' \in G/H^1(K)$ where the basis elements of $G/H^1$ contributing in $g$ all precede, or coincide with the minimal of, the ones contributing in $g'$. To prove that this property does not depend on the order of the basiselements which contribute, we need equation (2.2). This equation lets us interchange elements which are in the wrong order, as it lets us use $(x \cdot \lambda y)_n = (\lambda y \cdot x[x, \lambda y])_n$ with $[x, \lambda y] = \lambda(\psi(x, y) - \psi(y, x))$ for $x, y \in G(\Phi)$, $\lambda \in K$. To be specific, we get that if $g \in V_i(K)$ then

$$((g_i, 0), (g, 0))_n = \sum_{a+b+2c=n} (g, 0)_a (g_i, 0)_b (0, \psi(g_i, g) - \psi(g, g_i))_{2c} = (g + g_i, \psi(g_i, g))_n,$$

where the first equality holds by seeing $g$ as a $K$-linear combination of $k$ elements of $G(\Phi)$ and interchanging $g_i$ with each of these $k$ elements one at a time and the fact that $H^1(K)_n$ commutes with $G(K)_m$ for each $n$ and $m$.

[3] Such a basis always exists for free modules, by the axiom of choice, though the existence is probably easier to prove from Zorns lemma. We note that on well-ordered sets transfinite induction applies, i.e. prove that a property $P$ holds for the minimal element 0 and then prove that if $P$ holds for all $g < h$, then $P$ also holds for $h$. We prove the transfinite induction principle. Suppose that $h$ is the minimal element such that $P$ does not hold. So, we know that $P$ holds for all $g < h$. Therefore, $P$ holds for $h$, a contradiction. So, there is no minimal element $h$ such that $P$ does not hold.

However, the proof of this proposition can be made to work without the axiom of choice by just doing induction on the number of basis elements contributing to an element. The difference is then, that you start with a not necessarily well-defined representation and prove that it is well-defined instead of starting with a well-defined representation and computing that the product coincides with what the product should be.

[4] The '$+\mu\psi(g, g')$' is part of the definition, to ease the formulation of the definition. As each sequence of $H^1(K)$ commutes with every sequence of $G(\Phi)$, we could divide the sequence $(0, \mu\psi(g, g'))_n$ out, to get an equivalent definition.

We can generalize this to general linear combinations, instead of a single $g_i$, by seeing the linear combination as a product
$$(\sum \lambda_i g_i, 0) = (\prod_i (\lambda_i g_i, 0)) \cdot (0, h),$$
which corresponds exactly to how we defined each of these coordinates.

Lastly, we observe that this proposition will also hold for potential sequence $\Phi$-groups which are not necessarily free if they are a quotient of a free sequence $\Phi$-group. This is always the case. ■

**Definition 2.2.7.** When we speak, as we were able to reduce the property sequence $\Phi$-group representation to one expressable in polynomial equations, about **sequence $\Phi$-groups**[5] we mean potential sequence $\Phi$-groups with a defining representation satisfying equations (2.1) and (2.2), or equivalently potential sequence $\Phi$-groups $G$ with a defining representation $\rho : G \longrightarrow A$ such that $\rho_K : G(K) \longrightarrow A \otimes K$ are also representations. For sequence $\Phi$-groups, it is only natural to just consider sequence $\Phi$-group representations.

*Remark* 2.2.8. In what follows, we will be interested in $G(\Phi[[s,t]])$ with representations in $A[[s,t]]$. This, technically[6], does not fall under the previous construction. So, we sketch why we can, without problem, speak about formal power series. Consider $\rho_G(\Phi[t]) \longrightarrow A \otimes \Phi[t]$. We can endow $A \otimes \Phi[t]$ with the metric $d(f,g) = 2^{-i}$ with $i$ degree in $t$ of the term with the lowest such degree in $f - g$. Then $A[[t]]$ is the completion of $A \otimes \Phi[t]$ with respect to this metric. We can also define $G(\Phi[[t]])$ using the same technique. There is a unique equicontinuous $\rho_{\Phi[[t]]}$ mapping $G(\Phi[[t]])$ to $A[[t]]$. Moreover, since all necessary conditions are polynomial equations, and since the metrics are chosen so that the representation, addition, multiplication and scalar multiplication are equicontinuous, we know that all conditions making use of only these operations will still hold for the unique equicontinuous extensions of these maps to the closures. Note that all conditions for the representation to be a sequence $\Phi$-group representation, are of that form.

**Definition 2.2.9.** The $(i,j)$-**linearization** of a sequence $\Phi$-group representation is
$$(x, z) \longmapsto x_i z_j.$$
It is also possible to define the $(k_1, \ldots, k_n)$-**linearization** inductively.

*Remark* 2.2.10. Suppose that $\rho : G \longrightarrow A$ is a sequence $\Phi$-group representation. Note that a polynomial identity on an algebra $A$, which is stated in $x_i$'s for $x \in G(\Phi)$ is satisfied strictly if and only if the polynomial identity and all its linearizations hold. Specifically, we can linearize a polynomial $p(x_1, \ldots, x_n)$ formed out of monomials $\prod_{(i,j)\in I} (x_i)_j$ for some ordered finite subset $I$ of $\mathbb{N}^+_{\leq n} \times \mathbb{N}$, by formally comparing terms belonging to $\lambda^j$ of
$$p(x_1, \ldots, x_i \cdot \lambda z_i, \ldots, x_n).$$
By applying recursion until everything is a multilinear map, we get all the linearizations. If all the linearizations of a polynomial identity hold, then the polynomial identity will hold strictly. One can prove this analogous to Proposition (1.6.3). Specifically, we already know that the statement is true for the restriction of the polynomials to $H^1$. So, if $(a,0)(c,h+t) = (a+c,h)$. We see that
$$p(a+c,h) = p(a) + p_1((a,0),(c,h+t)) + \ldots + p(c,h+t),$$

[5] We will in what comes not make reference to the fact that the sequence groups have class 2, since that would be repetitive verbiage which does not provide additional information.

[6] In $A \otimes \Phi[[s,t]]$ we only allow finite linear combinations of terms $a \otimes f$, whilst we do want infinite linear combinations of terms $f_a \cdot a$, though only finitely many for each different monomial $s^i t^j$.

which, if applied to all linearizations of $p$ proves that all linearizations hold for bigger linear combinations. Therefore, we can use the fact that all linearizations hold as a substitute for the fact that such a polynomial equation holds strictly.

*Remark* 2.2.11. If $1/2 \in \Phi$, it is possible to reparametrize a sequence $\Phi$-group to be in some standard form. Suppose $G$ is a sequence $\Phi$-group over $\Phi$ with defining representation in $A$. Specifically, let $x = (1, x_1, x_2, \ldots) \in G(K)$, then

$$x(-x) = (1, 0, 2x_2 - x_1^2, \ldots) \in H^1(K).$$

As such, $y = (1, 0, x_1^2/2 - x_2, \ldots)$ is an element of $H^1(K)$. We compute that

$$xy = (1, x_1, x_1^2/2, \ldots).$$

We say that a sequence $\Phi$-group over a ring $\Phi$ with $1/2$ is in **standard form** if it satisfies

$$(x, 0) \longmapsto (1, x_1, x_1^2/2, \ldots).$$

We note that, if reparametrized, the sequence group has operation $(a, b)(c, d) = (a + c, b + d + [a, c]/2)$, where $[a, c]$ can either be seen as the commutator in the group, or as the Lie commutator in any defining representation.

## 2.3 An essential theorem

In this section, we formulate a certain theorem that will be useful in many different ways. Firstly, it will allow us to construct some divided power series out of specific families of elements in a Hopf algebra. At the same time, it can be used to construct exponentials in Lie algebras from similar families of elements. Lastly, the elements of algebras defined by the theorem will be used to define sequence pairs and will play a very important role throughout this thesis. To prove all these things at the same time, we will need a fairly general formulation of the theorem. We call it a fairly general formulation, as it is, in essence, a theorem about divided power series in Hopf algebras.

To prove the theorem of this section, we first prove the theorem in a very special case. Suppose that $Z$ is a free unital associative $\mathbb{Z}$-algebra generated by the elements $(a, b)$ for $a, b \in \mathbb{N}$, $b > 0$. We also use $(a, 0)$ to denote 0 for $a > 0$ and set $(0, 0) = 1$. We consider the unique morphism on $Z$ defined by

$$\Delta(a, b) = \sum_{\substack{i+j=a \\ k+l=b}} (i, k) \otimes (j, l).$$

*Remark* 2.3.1. We could also define a counit $\epsilon : Z \longrightarrow \mathbb{Z}$ by setting $\epsilon(a, b) = 0$ for $a \neq 0 \neq v$ and $\epsilon(1) = 1$. With this counit, $Z$, together with $\Delta$ and $\epsilon$, forms bialgebra. However, we will not need this bialgebra structure. In the current description of $Z$ it is not obvious that we could endow it with an antipode. However, the following lemma shows that $Z$ is generated by elements which are elements of divided power series $(1, [a, b], [2a, 2b], \ldots)$. For these elements it is easier to compute the antipode. Specifically, if one computes the inverse of the power series $(\sum t^n[na, nb])^{-1} = \sum t^n[na, nb]'$, then one gets the antipode by setting $S[na, nb] = [na, nb]'$.

**Lemma 2.3.2.** *There exist elements* $[a, b] \in Z$ *for* $a, b \in \mathbb{N}$, $b > 0$ *such that*

$$(n, m) = \sum_{t \in U(n,m)} t,$$

*with*

$$U(n,m) = \{[a_1,b_1]\dots[a_k,b_k] | a_1/b_1 < a_2/b_2 < \dots < a_k/b_k, \sum a_i = n, \sum b_j = m\},$$

*where, for each* $a, b$ *with gcd*$(a, b)$ *= 1 and each* $n \in \mathbb{N}$, *we have*

$$\Delta[na, nb] = \sum_{i+j=n} [ia, ib] \otimes [ja, jb],$$

*i.e.* $(1, [a, b], [2a, 2b], \dots)$ *is a divided power series.*

*Proof.* We prove, by induction on $b$, that for

$$[a, b] := (a, b) - \sum_{t \in U(a,b), t \neq [a,b]} t,$$

the definition of $\Delta$ coincides with the action of $\Delta$ indicated in the formulation of the lemma. Note that $\Delta$ and $[a, b]$ are well defined through recursion on the right hand side, as $[a, b]$ is the only term in $U(a, b)$ which has a contribution $[\cdot, b]$ in it.

For $b = 1$ we get the induction basis, as $(a, 1) = [a, 1]$ and

$$\Delta(a, b) = (a, 1) \otimes 1 + 1 \otimes (a, 1).$$

Now, we determine

$$\Delta(a, b) - \Delta(\sum_{\substack{t \in U(a,b) \\ t \neq [a,b]}} t).$$

We know, by induction, that

$$\Delta(a, b) = (a, b) \otimes 1 + 1 \otimes (a, b) + \sum_{\substack{i+j=a \\ k+l=b \\ kl \neq 0}} \sum_{(t,t') \in U(i,k) \times U(j,l)} t \otimes t' = \sum_{\substack{i+j=a \\ k+l=b \\ (t,t') \in U(i,k) \times U(j,l)}} t \otimes t'.$$

We try to identify which terms in

$$\sum_{\substack{i+j=a \\ k+l=b}} \sum_{(t,t') \in U(i,k) \times U(j,l)} t \otimes t'$$

cancel out with terms in

$$-\Delta(\sum_{\substack{t \in U(a,b) \\ t \neq [a,b]}} t).$$

To achieve that, we define $\psi_{ijkl} : U(i, k) \times U(j, l) \longrightarrow U(i + j, k + l)$ for all $i, j, k, l$. The purpose of this $\psi$ is to encode the following relation: $\psi(t, t') = t''$ if and only if $t \otimes t'$ is a term in $\Delta(t'')$. We will need the induction hypothesis to prove that the we $\psi$ construct has the right properties. The functions $\psi_{ijkl}$ will already map all the terms in $U(i, k) \times U(j, l)$ which do not come from any term of $U(a, b) \setminus \{[a, b]\}$ to $[a, b]$. As this $\psi$ will encode the asked relation on all the $t'' \neq [a, b]$, we get that $\psi$ necessarily encodes the relation as well for $[a, b]$. So, now we give the construction of $\psi$. We put

$$\psi_{00kl}(1, a) = a,$$

$$\psi_{ij00}(a, 1) = a,$$

$$\psi_{ijkl}([a_0,b_0]r_0,[a_1,b_1]r_1) = \begin{cases} [a_0+a_1,b_0+b_1]\psi_{i-a_0,j-b_0,k-a_1,l-b_1}(r_0,r_1) & a_0/b_0 = a_1/b_1 \\ [a_0,b_0]\psi_{i-a_0,j-b_0,k,l}(r_0,[a_1,b_1]r_1) & a_0/b_0 < a_1/b_1 \\ [a_1,b_1]\psi_{i,j,k-a_1,l-b_1}([a_0,b_0]r_0,r_1) & a_0/b_0 > a_1/b_1 \end{cases}.$$

One easily sees that this $\psi$ encodes the asked relation for $t \in U(a,b), t \neq [a,b]$ using the induction hypothesis. The only terms which it maps to $[a,b]$ are exactly the terms of the form $[c,d] \otimes [e,f]$ with $a/b = c/d = e/f$, which is what we had to prove. ■

Now, we consider unital associative algebras $A$, $B$ and $C$, and a relation $\Delta$ between $A$ and $B \otimes C$, which we denote using $a\Delta \sum_i b_i \otimes c_i$ for $a \in A, b_i \in B, c_i \in C$. There is one requirement for this relation, namely that $a\Delta f$,$b\Delta g$ imply both $a+b\Delta f+g$ and $ab\Delta fg$. We consider families of elements $(a,b)$ for $a,b \in \mathbb{N}$ such that

$$(a,b) \quad \Delta \sum_{\substack{i+j=a \\ k+l=b}} (i,k) \otimes (j,l)$$

and $(a,0) = \delta_{a0}$.

**Theorem 2.3.3.** *Suppose that $\Delta$ is a relation satisfying the two aforementioned properties. Let the $(a,b)$ be three families (in $A$, $B$ and $C$) of elements satisfying the mentioned relations. Each $(n,m)$ (in $A$, $B$ and $C$) can be written as*

$$(n,m) = \sum_{t \in U(n,m)} t,$$

*with*

$$U(n,m) = \{[a_1,b_1]\dots[a_k,b_k] | a_1/b_1 < a_2/b_2 < \dots < a_k/b_k, \sum a_i = n, \sum b_j = m\},$$

*such that, for each $a,b$ with gcd($a,b$) = 1 and each $n \in \mathbb{N}$, the following relation holds*

$$[na,nb] \quad \Delta \sum_{i+j=n} [ia,ib] \otimes [ja,jb].$$

*Proof.* Lemma (2.3.2) proves this theorem in the special case of $A = B = C = Z$ and $x \; \Delta \; y$ if $\Delta(x) = y$. Now, we consider the unique morphisms $\phi_A, \phi_B, \phi_C$ going from $Z$ to $A, B$ and $C$ determined by the fact that they map

$$(a,b) \longrightarrow (a,b).$$

Observe that $\Delta(z) = \sum_i b_i \otimes c_i$ implies that

$$\phi_A(z) \quad \Delta \quad \sum_i \phi_B(b_i) \otimes \phi_C(c_i),$$

because $z$ is a polynomial in the $(a,b)$ and $\sum_i b_i \otimes c_i$ is the unique element of $Z \otimes Z$ formed by applying $\Delta$ on certain elements $(a,b)$ and the using the compatibility of $\Delta$ with sum and multiplication. ■

**Definition 2.3.4.** Given families satisfying the conditions of Theorem (2.3.3), we will, for coprime $a$ and $b$, sometimes refer to the whole sequence $(1,[a,b],[2a,2b],\dots)$ as $\exp[a,b]$. Sometimes we will use the same notation to, instead, denote the sum $1+[a,b]+[2a,2b]+\cdots$. However, the second use is only permitted if this is a finite sum, or if we add a scalar which turns it into a formal power series. Furthermore, the distinction between those use cases should be obvious from the context.

Using the exponentials (with $\exp s[a,b] = 1 + s[a,b] + s^2[2a,2b],\ldots$), we could reformulate what we proved as

$$\sum s^a t^b (a,b) = \prod_{(a,b)\in S} \exp(s^a t^b [a,b]),$$

with $S$ the set of coprime $(a,b)$ and order $(a,b) < (c,d)$ if $a/b < c/d$. Moreover, in what follows the elements $(a,b)$ will often be defined from

$$\exp(sx)\exp(ty)\exp(sx)^{-1} = \sum s^a t^b (a,b),$$

for some elements contained in certain sequence groups $x, y$.

## 2.4 Sequence pairs

### 2.4.1 Definition of sequence pairs

We consider sequence $\Phi$-groups and introduce sequence pairs. We recall that we assumed that the sequence group representations of sequence $\Phi$-groups are even.

*Motivation* 2.4.1. We try to motivate the forthcoming definition of sequence pairs. We want to consider pairs of sequence $\Phi$-groups such that there is a sort of pairing between them, using the adjoint representation. Specifically, let $A$ be an associative unital $\Phi$-algebra and $G_+, G_-$ be two sequence $\Phi$-groups in $A$. Let $x \in G_\pm(K)$ and $y \in G_\mp(K)$, we set

$$f(m,n) = (\mathrm{ad}_x^{(0)}(y_0), \mathrm{ad}_x^{(m)}(y_n), \ldots, \mathrm{ad}_x^{(mk)}(y_{nk}), \ldots).$$

We want that $f(m,n) = (1) \in G_\pm(K)$ for $m > 3n$, $f(3,1) \in H^1_\pm(K)$ and that $f(2,1) \in G_\pm(K)$ modulo some error terms contained the ideal generated by the $f(3,1)_i, i > 0$. We also want that for $m,n$ such that $3n > m > 2n$ that $f(m,n) = (1)$ modulo some error terms contained in the ideal generated by the $f(3,1)_i, i > 0$. However, the indication that some equalities must hold modulo some ideal is not optimal. An exact description of the error terms is much better. This exact description is possible using the elements $[a,b]$ introduced in Theorem (2.3.3).

Intuitively, this generalizes Faulkners [Fau00] approach to divided power representations of Jordan pairs. Specifically, he asks that $f(m,n) = (1)$ for all $m > 2n$ and that $f(2,1) \in G_\pm(K)$. We see that this restriction holds for elements $x, y$ in a sequence pair if and only $f(3,1) = (1)$, which will definitely be true for any sequence pair representation of a Jordan pair.

Suppose that $G_+, G_-$ are sequence $\Phi$-groups with a defining representation in $A$. We define, for $x \in G_\pm(K)$, $y \in G_\pm(K)$, and $a, b \in \mathbb{N}$, the elements

$$(a,b) = \mathrm{ad}_x^{(a)}(y_b).$$

Note that $(a,0) = \delta_{a0}$. Now, we want to define some other elements from these $(a,b)$ using the definition of elements $[a,b]$ as in Theorem (2.3.3). We recall that

$$U(n,m) = \{[a_1,b_1]\ldots[a_k,b_k] | a_1/b_1 < a_2/b_2 < \ldots < a_k/b_k, \sum a_i = n, \sum b_j = m\},$$

for symbols $[a,b]$ with $0 < a,b \in \mathbb{N}$, we also set $[a,0] = \delta_{a0}$. For a family of elements $(a,b)$ in any algebra $B$, we can interpret these $[a,b]$ as elements of $B$, using

$$(a,b) = \sum_{t\in U(a,b)} t,$$

which gives us the recursive formula

$$[a,b]=(a,b)-\sum_{\substack{t\in U(a,b)\\ t\neq[a,b]}} t.$$

Note that there is no relation $\Delta$, yet[7], upon which we can apply Theorem (2.3.3) to get nice properties for the elements $[a,b]$. Nonetheless, any application of that theorem on the family $(a,b)$ will necessarily give the elements $[a,b]$.

**Definition 2.4.2.** We assume, in the following definition, that $K$ is a free $\Phi$-**alg**-variable, in order to prevent stating repeatedly that all statements must hold for all $K\in\Phi$-**alg**. Consider a pair $G=(G_-,G_+)$ of sequence $\Phi$-groups in $A$. Suppose that, for $\sigma=\pm$, there exist operators

$$\begin{aligned} Q_\sigma(K):G_\sigma(K)&\longrightarrow \mathrm{Hom}_{Set}(G_{-\sigma}(K),G_\sigma(K)),\\ T_\sigma(K):G_\sigma(K)&\longrightarrow \mathrm{Hom}_{Set}(G_{-\sigma}(K),H^1_\sigma(K)).\end{aligned}$$

We assume that these, for $x\in G_\sigma(K), y\in G_{-\sigma}(K)$, satisfy

$$[3n,n]=T_\sigma(K)(x)(y)_{2n}, \tag{2.3}$$
$$[2n,n]=Q_\sigma(K)(x)(y)_n, \tag{2.4}$$

with $[a,b]$ and $U(a,b)$ as introduced just before the definition. If, in addition,

$$\mathrm{ad}_x^{(n)}(y_m)=0 \quad \text{for } n>3m \tag{2.5}$$

and

$$\mathrm{ad}_x^{(a)}(y_b)=\sum_{t\in\tilde U(a,b)} t, \tag{2.6}$$

for all $a,b$ such that $3b>a>2b$, with

$$\tilde U(a,b)=\{t\in U(a,b)|t=i[3j,j]\text{ for some } j>0, i\in U(a-3j,b-j)\},$$

hold, then we call $G$ a **sequence pair**. Note that there cannot be a contribution of $[3j,j]$ with $j>0$ in the $i$ in $\tilde U(a,b)$. A **homomorphism** $\rho:G\longrightarrow G'$ of sequence pairs, is a pair of sequence group morphisms $\rho_\sigma:G_\sigma\longrightarrow G'_\sigma$ such that $Q_\sigma(K)(\rho_\sigma(x))(\rho_{-\sigma}(y))=\rho_\sigma(Q_\sigma(K)(x)(y))$ and $T_\sigma(K)(\rho_\sigma(x))(\rho_{-\sigma}(y))=\rho_\sigma(T_\sigma(K)(x)(y))$ for all $\sigma=\pm$, $x\in G_\sigma(K), y\in G_{-\sigma}(K)$. A **sequence pair representation** is a pair of sequence group representations so that, firstly, the image forms a sequence pair $P$ and, secondly, the representations induce a morphism onto $P$. Suppose $\rho$ is a sequence pair representation formed by faithful sequence group representations, then we call $\rho$ a **defining representation**.

*Remark* 2.4.3.

- The name defining representation indicates that if we forget the sequence pair structure and only have a pair of sequence group representations corresponding to a defining representation, then we can recover, i.e. define, the operators $Q$ and $T$. Note, however, that we need to require that the defining representation was a sequence pair representation and not just a pair of sequence group representations such that their images form a sequence pair, as this would not necessarily yield the same $Q$ and $T$.

[7] Eventually we will see that these elements actually do, in some sense, come from an application of Theorem (2.3.3) in a Hopf algebra, namely the universal representation. In fact, we could have first defined what we call weak sequence pairs (cf. Definition (2.4.4)), then realize that there is an associated Hopf algebra in which we can apply the theorem to the family $(a,b)$, and then restrict ourselves to a specific subset of weak sequence pairs.

- We will never write $Q_\sigma(K)$ again, and just write $Q_\sigma$. The same is true for $T$. In what follows, we will often write $Q_x(y)$ instead of $Q_\sigma(x)(y)$ if the signs are obvious from, or do not matter in the context. Moreover, we will often use $\sigma$ to denote a sign $\pm$ without always stating that $\sigma = \pm$.

- We will often write, "let $x \in G_\sigma(K)$" and use $K$ as a free $\Phi$-**alg** variable, instead of first fixing $K \in \Phi$-**alg**. This reduces the verbosity, as first fixing $K$ while you functionally use it as a free $\Phi$-**alg** variable does not tell anything. Furthermore, if it is used to denote $G_\sigma(K)$ there is no doubt that $K$ is an element of $\Phi$-**alg**.

- The category-theoretic notions of mono-, epi- and isomorphism coincide with the fact that the underlying morphisms $G_\sigma(\Phi) \longrightarrow G'_\sigma(\Phi)$, as abstract groups, is a mono-, epi- or isomorfism.

- All the expressions involved in the definition are linearizable. Most expressions fall under Remark (2.2.10). For $Q_\sigma, T_\sigma$, we need to apply a little trick. They correspond to polynomials in a defining representation. Therefore, they are linearizable in the defining representation. Note that these operators can be fully reconstructed, seen as polynomials in a defining representation, from the restrictions $Q_\sigma(\Phi), T_\sigma(\Phi)$ and all linearizations. A consequence is that we are perfectly justified to speak about the identities of Definition (2.4.2) and all their linearizations. Moreover, if $G$ is a sequence pair, and if there is a pair of sequence group representations which satisfy Definition (2.4.2), but only for $\Phi$, and all linearizations of the identities are satisfied over $\Phi$, then it is a sequence pair representation.

- One easily sees that $T$ is of the form
$$T_\sigma(x)(y_1, y_2) = (0, f_x(y_1)),$$
with $f_x(y_1)$ linear in $y_1$. So, $T_\sigma(x)$ is a group homomorphism factoring through $G/H^1_\sigma(K)$. For $Q$, it is not that easy. We can split $Q$ into two parts, we look at
$$Q^1_\sigma : G_\sigma \longrightarrow \mathrm{Hom}_{Grp}(G/H^1_{-\sigma}, G/H^1_\sigma),$$
and
$$Q^2_\sigma : G_\sigma \longrightarrow \mathrm{Hom}_{Grp}(H^1_{-\sigma}, H^1_\sigma),$$
defined by
$$Q^1_\sigma(x)(y_1, \cdot) = zH^1_\sigma(K),$$
with the unique $zH^1_\sigma(K)$ such that $z_1 = [2,1]$ in a defining representation, and $Q^2_\sigma$ defined by just taking the restriction of $Q_\sigma(x)$ to $H^1(K)$. The fact that the image of $Q^2_\sigma$ also lies in a $H^1_\sigma(K)$ is a consequence of the fact that all representations are assumed to be even, so that
$$\mathrm{ad}_x^{(2n)}(y_n) = \begin{cases} 0 & \text{if } n \text{ odd} \\ [2n, n] & \text{always} \end{cases}, \quad \text{so} \quad (1, 0, [4,2], 0, [8,4], \ldots) \in H^1_\sigma(K),$$
for $x \in G_\sigma(K)$ and $y \in H^1_{-\sigma}(K)$. Note that $Q^1_\sigma$, $Q^2_\sigma$ fully[8] determine $Q_\sigma$ and vice versa. We will sometimes use these specific operators instead of the full $Q_\sigma$ if we want to do explicit computations.

[8] Only if we know that $G$ forms with these operators a sequence pair. Otherwise, we do not know whether $Q$ maps into $G$.

- The binomial divided power representations of Jordan pairs of [Fau00] correspond exactly with the sequence pairs such that $H^1 = [G, G] = 0$. Specifically, if $G$ is such a sequence pair, then $G = (V^+, V^-)$ as groups and the operators $Q$ makes $G$ into a Jordan pair. We prove this in Proposition (4.1.15). Conversely, the restriction that $\mathrm{ad}_x^{(2n)}(y_n) = Q_x(y)_n$ and $\mathrm{ad}_x^{(n)} y_m = 0$ for $n > 2m$ for binomial divided power representations implies all the identities of Definition (2.4.2). So, the binomial divided power representations of Jordan pairs are sequence pair representations.

- We will be working a lot with the elements $(a, b)$ and $[a, b]$ as introduced before the definition. These elements are only defined once you fix $x \in G_\sigma(K)$ and $y \in G_{-\sigma}(K)$. So, if we use these elements we will always indicate which $x$ and $y$ we use. However, we will not repeat endlessly that $(a, b) = \mathrm{ad}_x^{(a)}(y_b)$ and that the $[a, b]$ are defined using recursion from that. So, we will use the notation $(a, b)$ and $[a, b]$ a lot, and consider them, sort of, as operators which are an integral part of each sequence pair representation.

**Definition 2.4.4.** Suppose that $G$ is a pair of sequence $\Phi$-groups which satisfy restrictions (2.3), (2.5), and have operators $Q^1, Q^2$ such that

$$Q_x^1(y)_1 = [2, 1], \qquad Q_x^2(h)_{2n} = [4n, 2n] = \mathrm{ad}_x^{(4n)}(h_{2n}),$$

for all $x \in G_\pm(K), y \in G_\mp(K)$, $h \in H^1_\mp(K)$ and $n \in \mathbb{N}$. If, additionally, restriction (2.6) holds for $x \in G_\pm(K)$ and $y \in H^1_\pm(K)$, then we call $G$ a **weak sequence pair**. The adaptation of restriction (2.6) is so that it expresses that $\mathrm{ad}_x^{(n)}(h_m) = 0$ for $n > 2m$ and $h \in H^1_\pm(K), x \in G_\mp(K)$. It asks nothing more and nothing less. The **weak sequence pair representations** are exactly the representations of the sequence $\Phi$-groups which satisfy these restrictions.

*Remark* 2.4.5.

- We can reformulate what weak sequence pairs are, only utilizing elements of the form $(\cdot, \cdot)$. Specifically, we ask, for $x \in G_\sigma(K), y \in G_{-\sigma}(K), h \in H^1_{-\sigma}(K), n \in \mathbb{N}$, that

$$\begin{aligned} T_x(y)_n &= (3n, n), \\ Q_x^1(y)_1 &= (2, 1), \\ Q_x^2(h)_n &= (4n, 2n). \end{aligned}$$

  We also ask that all $(a, b) = 0$ with $a > 3b$ for $x$ and $y$, and that all $(a, b) = 0$ with $a > 2b$ for $x$ and $h$.

- We will not use weak sequence pairs that much. Nevertheless, they are conceptually useful if we were not yet able to prove the existence of a $Q$ with nice properties. Specifically, if $1/2 \notin \Phi$ this replacement will save us some trouble. Later, we will be able to prove that, if $1/6 \in \Phi$, each weak Kantor-like (Definition (4.2.1)) sequence pair is a sequence pair. The same is true for Jordan-Kantor-like sequence pairs. The question if each weak sequence pair representation is a sequence pair representation is of a more subtle kind.

- Applied to Jordan pairs, the weak sequence pairs with $H^1_\pm = 0$ would be Jordan pairs if $1/2 \in \Phi$. If $1/2 \notin \Phi$, then these notions do not coincide. Specifically,

$$Q_x Q_y Q_x = Q_{Q_x y}$$

  does not necessarily hold.

**Lemma 2.4.6.** *Suppose $G$ is a pair of sequence groups in $A$. Conditions (2.5) and (2.6) are equivalent to $[a, b] = 0$ if $a/b \neq 3$ and $a > 2b$.*

*Proof.* Suppose conditions (2.5) and (2.6) hold. We prove that $[a,b]=0$ for all $a>3b$. Note that $U(a,b)$ consists entirely out of elements

$$[a_1,b_1]\dots[a_n,b_n],$$

with $a_i/b_i$ strictly increasing, such that $\sum a_i=a$ and $\sum b_i=b$. As a consequence, we can infer from $a>3b$ that $a_n>3b_n$. So, we can apply induction, since the theorem holds for $b=0$, with induction step $[a,b]=0-\sum_{t\in U(a,b),t\neq[a,b]}0$. Analogously, we prove that $[a,b]=0$ for all $3b>a>2b$. Specifically, $\tilde{U}(a,b)$ contains by induction the subset of $U(a,b)\setminus\{[a,b]\}$ of elements which are non zero in $A$. From (2.6), we conclude that

$$[a,b]=\mathrm{ad}_x^{(a)}(y_b)-\sum_{t\in\tilde{U}(a,b)}t=0.$$

The converse holds since

$$\mathrm{ad}_x^{(a)}(y_b)=\sum_{t\in U(a,b)}[a,b]$$

holds by definition. To be precise, if $a>3b$, then each

$$[a_1,b_1]\dots[a_n,b_n]\in U(a,b)$$

is zero since $a_n>3b_n$. If, $a>2b$ then the same still applies, but each term $t=[a_1,b_1]\cdots[a_n,b_n]\in U(a,b)$ can only contribute if $a_n>2b_n$ and $b_n>0$, so that we get that the terms $t\in U(a,b)$ which are non-zero are necessarily contained in $\tilde{U}(a,b)$, as they should end on a $[3j,j]$ with $j\neq 0$. ∎

*Remark* 2.4.7. We will often use Lemma (2.4.6) to prove that conditions (2.5) and (2.6) hold. In the current formulation, it might even be more natural to use the equivalent description of the lemma. However, the only conceptual difference is in condition (2.6). This formulation of the condition has the advantage that it nicely indicates that $(a,b)=0$ for $a>2b$ if we divide out by the ideal generated by the $[3i,i]$ for $i>0$.

### 2.4.2 From Jordan-Kantorpairs to sequence pairs

We shall prove, in this subsection, that the representations of Example (2.1.8) are actually a sequence pair representation if $1/5\in\Phi$. In the $1/5\notin\Phi$ case we give a sufficient condition (we later prove this condition is necessary) for these representations to form a sequence pair representation.

**Theorem 2.4.8.** *Let $P$ be a Jordan-Kantor pair. The representations of Example (2.1.8) in the endomorphism algebra of $L=\mathit{TKK}(P,\textbf{\textit{InStr}}(P)+\Phi\zeta)$ with $\zeta$ a grading element, form a sequence pair representation, which we call the* ***TKK representation*** *of the sequence pair associated to the Jordan-Kantor pair, if either*

- $1/5\in\Phi$,
- $x_n[a,b]=\sum_{i+j=n}[x_i a,x_j b]$ *for all* $a,b\in L$ *and* $n\in\mathbb{N}$, $x\in G_\sigma(\Phi)$.

*Remark* 2.4.9.
- One can reformulate the second condition of the either-statement to be that, with $(n,m)=\mathrm{ad}_x^{(n)}(y_m)$, $(5,1)=0$ for all $x\in G_\sigma(\Phi),y\in G_{-\sigma}(\Phi)$ and $(5,2),(6,2)=0$ if $y\in H^1_{-\sigma}(\Phi)$. These conditions are very reminiscent of the axiom $Q_xD_{y,x}=D_{x,y}Q_x$ for Jordan pairs (as this axiom is equivalent with $(3,1)=0$).

- It is interesting to note that all central simple structurable algebras over fields with characteristic $5$ will, by the previous theorem, induce sequence pair representations. Specifically, Stavrova [Sta20, Theorem 1.1 and Theorem 2.1] showed that $x_n[a,b] = \sum_{i+j=n}[x_i a, x_j b]$ for all $n \in \mathbb{N}, a, b \in L$.
- The condition $x_n[a,b] = \sum_{i+j=n}[x_i a, x_j b]$ is also necessary. For Kantor pairs, we note this in Remark (4.2.9). For Jordan-Kantor pairs this property is necessary for the same reasons.
- It is worth noting that the linear and quadratic Jordan algebras are the same if $1/2 \in \Phi$, but that the linear Jordan pairs (= Kantor pairs so that the standard embedding in a Lie algebra is a 3-graded Lie algebra) are not equivalent to quadratic Jordan pairs if $1/3 \notin \Phi$. So, we are probably not considering all Kantor pairs if $1/5 \notin \Phi$.

We will already refer in the following lemmas to this pair of representations, as the TKK representation. At this moment we only know that it is a pair of sequence group representations.

**Lemma 2.4.10.** *The TKK representation is a pair of sequence $\Phi$-group representations.*

*Proof.* It is straightforward to see that there are sequence group representations

$$G_\sigma(K) \longrightarrow \mathrm{End}_K(L \otimes K).$$

However, this does not necessarily imply that there is a sequence $\Phi$-group representation

$$G_\sigma(K) \longrightarrow \mathrm{End}_\Phi(L) \otimes K.$$

We note that it is sufficient to argue that equations (2.1) and (2.2) hold, to prove that we have a sequence $\Phi$-group representation. We could use direct computation to prove that. Nevertheless, we will argue differently. We want to conclude that $\lambda^n \cdot g_n = (\lambda \cdot g)_n$, and $(gh)_n = \sum_{i+j=n} g_i h_j$, are satisfied strictly. We will argue that this is the case using linearizations. However, we should be careful and only linearize in well-defined sequence group representations. Note that all linearizations of a polynomial identity hold, if and only if the polynomial identity holds over all $\Phi[t]/(t^n)$, as a polynomial of homogeneous degree $n$ can be linearized over $\Phi[t]/(t^{n+1})$. We have sequence group representations

$$G_\sigma(\Phi[t]/t^n) \longrightarrow \mathrm{End}_{\Phi[t]/(t^n)}(L \otimes \Phi[t]/(t^n)) \cong \mathrm{End}_\Phi(L) \otimes \Phi[t]/(t^n),$$

with the isomorphism because $\Phi[t]/(t^n)$ is a finite free $\Phi$-module. So, all linearizations of the polynomial identities necessarily hold. In particular, equations (2.1) and (2.2) hold. So, we have a sequence $\Phi$-group representation. ■

**Lemma 2.4.11.** *Consider a $5$-graded Lie algebra $L$ with grading element over $\Phi$. Suppose that $D$ is a derivation on $L$ such that there exists a $j \neq 0$ so that $DL_i \subseteq L_{i+j}$ for the grading components $L_i$ of $L$. If $1/6 \in \Phi$, then $D$ is an inner derivation.*

*Proof.* Suppose that $\zeta$ is the grading element and $x \in L_i$. We compute

$$iDx = D[\zeta, x] = [D\zeta, x] + [\zeta, Dx] = [D\zeta, x] + (i+j)Dx.$$

Since $j \neq 0$ and $1/j \in \Phi$, we get

$$Dx = \frac{-1}{j}[D\zeta, x].$$

Hence, $D$ is an inner derivation. ■

**Definition 2.4.12.** A **Lie-exponential** is an endomorphism $1 + E_1 + E_2 + \ldots + E_n$ of a Lie algebra $L$ such that $E_1$ is a derivation, $E_2[a, b] = [E_2 a, b] + [E_1 a, E_1 b] + [a, E_2 b]$, for all $a, b \in L$, etc. We note that these are not necessarily automorphisms. Furthermore, infinite sums $1 + E_1 + \ldots$ which are well defined and which are Lie exponentials for each $S_n = 1 + E_1 + \ldots + E_n$, are also called Lie exponentials.

**Lemma 2.4.13.** *Suppose $1/6 \in \Phi$. Each Lie-exponential $E = 1 + E_1 + E_2 + E_3 + E_4$ on a Lie 5-graded Lie algebra $L$ with grading element, such that there exists $j = \pm 1$ so that $E_i(L_k) \subset L_{ij+k}$ is of the form*

$$\exp(x_1 + x_2) = 1 + \sum_{i=1}^{4} \sum_{j+2k=i} \frac{(ad\, x_1)^j}{j!} \frac{(ad\, x_2)^k}{k!},$$

*for some $(x_1, x_2) \in L_\sigma \times L_{2\sigma}$, $\sigma = \pm$.*

*Proof.* Lemma (2.4.11) shows that all derivations with a certain action on the grading are inner are inner. Hence, for $E$ we know that $E_1$ is an inner derivation. Therefore, we get $E_1 = \text{ad } x_1$ for some $x_1 \in L_\sigma$.

Now we use the fact that $2E_2 - E_1^2, 3E_3 - 3E_1E_2 - E_1^3$ and a similar expression for $E_4$, namely $P_4 = 4E_4 - 2E_2^2 + E_1E_3 - 2E_3E_1 + E_2E_1^2$, are derivations[9]. For $2E_2 - E_1^2$ and $3E_3 - 3E_1E_2 - E_1^3$ this is an easy computation. For $P_4$, this is still straightforward, but slightly more involved.

As a consequence, we see that $2\text{ad } x_2 = 2E_2 - E_1^2$ and that $0 = 3E_3 - 3E_1E_2 - E_1^3$. So, $E_3$ and $E_4$ are uniquely determined from $E_1$, $E_2$ and these equations. One easily sees that $\exp(x_1 + x_2)$ satisfies exactly the same properties. ∎

**Lemma 2.4.14.** *For the TKK representation $\rho$ of $G$ in End$(L)$ the equation*

$$x_n \cdot [a, b] = \sum_{i+j=n} [x_i \cdot a, x_j \cdot b]$$

*holds, for each $x \in G_\pm(\Phi)$, $a, b \in L$ and for $n \leq 4$. Moreover, if $1/5 \in \Phi$ this holds all $n$ with $x_k = 0$ for $k > 4$.*

*Proof.* Suppose $x = (c, d) \in L_\sigma \oplus L_{2\sigma}$, then

$$\rho(x) = (1, \text{ad } c, (\text{ad } c)^2/2, (\text{ad } c)^3/6, (\text{ad } c)^4/24) \times (1, 0, \text{ad } d, 0, (\text{ad } d)^2/2).$$

For $C = (1, \text{ad } c, (\text{ad } c)^2/2, (\text{ad } c)^3/6, (\text{ad } c)^4/24)$ and $D = (1, 0, \text{ad } d, 0, (\text{ad } d)^2/2)$, one easily shows, using induction, that the lemma holds (using, for the moreover part, that $(\text{ad } c)^5/(5!) = 0 = (\text{ad } c)^6/(6!)$ and the fact that $[D_i a, D_j b] = 0$ for all $a, b \in L$ if $D_k$ is $k$ graded and if $i + j \geq 7$). Now, notice that

$$\sum_{i+j=k} C_i D_j [a, b] = \sum_{i+l+m=k} C_i [D_l a, D_m b] = \sum_{o+p+l+m} [C_o D_l a, C_p D_m b]$$

holds for all $k$. This proves that $x$ satisfies the lemma. ∎

[9] These are important expressions. These are primitive for any divided power series, which is in some sense equivalent to saying that these are derivations in this context. Moreover, it is actually possible, cf. Lemma (4.2.10), to show that the expressions with $E_3$ and $E_4$ are 0 for each sequence $\Phi$-group representation.

*Remark* 2.4.15. Note that the result of the previous lemma definitely holds for all $x \in H^1(K)$, even without the assumption $1/5 \in \Phi$. We can easily extend the lemma to prove that

$$x_n \cdot [a, b] = \sum_{i+j=n} [x_i \cdot a, x_j \cdot b]$$

holds for each $x \in G_\pm(K)$, if it holds for all $y \in G_\pm(\Phi)$, by using that each such $x$ can be written as a product of $K$-scalar multiples of such $y$.

**Lemma 2.4.16.** *Suppose that*

$$x_n \cdot [a, b] = \sum_{i+j=n} [x_i \cdot a, x_j \cdot b]$$

*holds, for each $x \in G_\pm(\Phi)$, $a, b \in L$ and for all $n$. For coprime $n$ and $m$, the element $1 + [n, m] + [2n, 2m] + [3n, 3m] + [4n, 4m]$ is a Lie-exponential.*

*Proof.* We want to use Theorem (2.3.3) on the relation

$$a \quad \Delta \quad \sum b_i \otimes c_i \qquad \Longleftrightarrow \qquad a[u, v] = \sum [b_i u, c_i v] \text{ for all } u, v \in L,$$

for $a, b_i, c_i$ in the endomorphism algebra of a Lie algebra $L$. This relation is compatible with sums and multiplications. So, need to determine whether the elements $(u, v) = \mathrm{ad}_x^{(u)}(y_v)$ satisfy

$$(u, v) \quad \Delta \sum_{\substack{i+j=u \\ k+l=v}} (i, k) \otimes (v, l),$$

as $[n, m]$, $[2n, 2m]$, etc. can be seen to be defined from these elements if we apply Theorem (2.3.3).

We compute

$$(i, j) = \sum_{a+b=i} x_a y_j (x^{-1})_b \quad \Delta \quad \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b \\ a+b=i \\ j_1+j_2=j}} x_{a_1} y_{j_1} (x^{-1})_{b_1} \otimes x_{a_2} y_{j_2} (x^{-1})_{b_2} = \sum_{\substack{i_1+i_2=i \\ j_1+j_2=j}} (i_1, j_1) \otimes (i_2, j_2),$$

making use of the assumption that $x_n \Delta \sum x_i \otimes x_j$ in the TKK representation. So, we conclude that the element $1 + [n, m] + \ldots$ is a Lie exponential. ■

*Proof of Theorem (2.4.8).* Lemma (2.4.14) ensures that the second either case always holds. So, we can use lemmas that have the second either case as an assumption. Furthermore, we know, by remark (2.4.15), that this assumption holds for all $K \in \Phi$-**alg** instead of only $\Phi$. This means that we can forget that we are working over all $K \in \Phi$-**alg** and just consider a particular representation $G(K) \longrightarrow \mathrm{End}_\Phi(L) \otimes K$. Despite the fact that $\mathrm{End}_\Phi(L) \otimes K$ is not necessarily isomorphic to $\mathrm{End}_K(L \otimes K)$, we know that there is an induced action of $\mathrm{End}_\Phi(L) \otimes K$ on $L \otimes K$. First, we prove that the representation in $\mathrm{End}_K(L \otimes K)$ has the operators. Lemmas (2.4.16) and (2.4.13) show that the sequences

$$(1, [2, 1], [4, 2], \ldots) \quad \text{and} \quad (1, [3, 1], [6, 2], \ldots),$$

are exponentials contained in $G_+(K)$ and $G_-(K)$. This endows the pair of sequence groups with some operators $Q, T$ which satisfy identities (2.3) and (2.4) strictly.

We note that all $[n, m]$, for $3 \neq n/m > 2$, are 0 by the grading, except $[4, 1], [5, 1], [5, 2], [7, 3]$ which are, by Lemma (2.4.16), derivations. So, they are 0 by Lemma (2.4.11), since there cannot be any inner derivations which act on the grading by $\pm 3, \pm 4$. From Lemma (2.4.6), we conclude that conditions (2.5) and (2.6) hold.

Now we realize that we have representations $G_\sigma(K) \longrightarrow \mathrm{End}_K(L \otimes K)$ which satisfy all conditions. Analogous to how we argued in Lemma (2.4.10)[10], we can show that this implies that $G_\sigma(\Phi)$ satisfies all conditions. Specifically, the fact that we have representations in all $\mathrm{End}_K(L \otimes K)$ means that all linearizations of all the restrictions will hold. So, we have sequence pair representations in $\mathrm{End}_\Phi(L) \otimes K$. ■

### 2.4.3 From Hopf algebras to sequence pairs

We define

$$A_x(y) = m \circ (\mathrm{Id} \otimes S) \circ \Delta(x)(y),$$

with $m(a \otimes b)(c) = acb$. Note that this corresponds roughly to $\mathrm{ad}_x^{(\cdot)}$ for sequence groups. Namely, suppose that $x \mapsto (1, x_1, \ldots)$ is a sequence group representation which maps elements to divided power series, in that case we get $A_{x_i} = \mathrm{ad}_x^{(i)}$. Later, we will be able to interpret $\mathrm{ad}_x^{(\cdot)}$ to actually correspond, in a strict sense, to $A$.

**Lemma 2.4.17.** *Let $H$ be a Hopf algebra, then $\Delta \circ S = \tau \circ S \otimes S \circ \Delta$ holds, with $\tau(a \otimes b) = b \otimes a$.*

*Proof.* A proof of this has been given by Abe and Sweedler [AS77, Theorem 2.1.4]. ■

**Lemma 2.4.18.** *Let $H$ be a Hopf algebra and suppose that $x$ and $y$ are divided power series. Set*

$$(n, m) := A_{x_n}(y_m),$$

*then $\Delta(n, m) = \sum_{\substack{i+j=n \\ k+l=m}} (i, k) \otimes (j, l)$.*

*Proof.* We compute

$$\begin{aligned}
\Delta(n, m) &= \Delta\Big(\sum_{i+j=n} x_i y_m S(x_i)\Big) \\
&= \sum_{\substack{a+b=i \\ c+d=m}} (x_a y_c \otimes x_b y_d) \Delta(S(x_i)).
\end{aligned}$$

By lemma (2.4.17), the statement follows. ■

**Corollary 2.4.19.** *Let $x$ and $y$ be divided power series in a Hopf algebra $A$. Suppose that $\epsilon(x_n) = 0$ for $n > 0$. There exist unique divided power series $1, [n, m], [2n, 2m], \ldots$, for all $n, m$ coprime, satisfying*

$$(n, m) = \sum_{t \in U(n,m)} t,$$

*with $U(n, m)$ the set of Theorem (2.3.3) and $(n, m) = A_{x_n}(y_m)$.*

*Proof.* This is Theorem (2.3.3) applied to the relation $a \Delta f$ if and only if $\Delta(a) = f$, on the family of elements $(n, m)$ (3 times the same family). This family has the right properties by Lemma (2.4.18) and the assumption $\epsilon(x_n) = 0$. Specifically, this assumption shows that $(n, 0) = \delta_{n0}$, as $A_{x_n}(1) = \eta(\epsilon(x_n)) = \delta_{n0}$. ■

[10] We need to be less careful, as we already know that we have sequence $\Phi$-group representations, so that it is definitely meaningful to speak about linearizations.

*Remark* 2.4.20. Consider in $H[[t]]$ the formal power series $e = \exp(t \cdot x) = \sum t^i x_i$, for a dps $x$ such that $\epsilon(x_i) = \delta_{i0}$. This is a group like element, as $\Delta(e) = e \otimes e$ and $\epsilon(e) = 1$ show. Therefore $e$ has an inverse $e^{-1}$ which does, necessarily, coincide with $S(e)$ since $1 = \epsilon(e) = (\mathrm{Id} \otimes S) \circ \Delta(e) = eS(e)$.

Thus, we can compute $a^{\exp(t \cdot x)} = \exp(t \cdot x)^{-1} a \exp(t \cdot x)$ for each such dps $x$ and $a \in H$. Note that the coefficient of $t^i$ equals $A_{S(x_i)}(a)$.

**Definition 2.4.21.** Suppose $H$ is a $\mathbb{Z}$-graded Hopf algebra where the primitive elements $P$ are 5-graded, i.e. $P_{-2} \oplus P_{-1} \oplus P_0 \oplus P_1 \oplus P_2$, compatible with the $\mathbb{Z}$-grading on the Hopf algebra. Then we call $H$ 2-**primitive** $\mathbb{Z}$-**graded**. Suppose that $x$ is a dps in $H$ such that $x_i \in H_i$ for each $i$, then we call $x$ a positive homogeneous dps. If $x_i \in H_{-i}$ for all $i$, we call it a negative homogeneous dps. Note that the exponentials of these divided power series are group like and, thus, invertible, as Remark (2.4.20) indicates.

**Lemma 2.4.22.** *Consider a cocommutative $\mathbb{Z}$-graded Hopf algebra $H$ over $\Phi$. The positive (resp. negative) homogeneous divided power series of $H$ form a sequence group. Moreover, if two homogeneous divided power series*

$$(1, x_1, \ldots, x_n, x_{n+1}, \ldots),\ (1, x_1, \ldots, x_n, x'_{n+1}, \ldots)$$

*coincide on the first $n$ elements and if $x_{n+1} \neq x'_{n+1}$, then $x_n - x'_{n+1}$ is primitive.*

*Proof.* Firstly, we note that divided power series are compatible with the scalar multiplication. Secondly, any easy computation shows that the group operation is internal. We only need to see that there are inverses. Recall that $\epsilon = 0$ on the non 0-graded parts. We use this to prove that there is an inverse, by computing

$$\delta_{n0} = \eta \circ \epsilon(x_n) = (\mathrm{Id} \otimes S) \circ \Delta(x_n) = \sum_{i+j=n} x_i S(x_j),$$

which means that $x \cdot S(x) = (1)$. The second statement is trivial. ∎

**Theorem 2.4.23.** *Suppose $H$ is a cocommutative 2-primitive $\mathbb{Z}$-graded Hopf algebra over $\Phi$. Suppose that for all $\pm 2$ graded primitive elements $x$ there exist an infinite homogeneous dps over $x$ and that, either*

- *$1/2 \in \Phi$ and for each primitive element which is $\pm 1$ graded, there exists an infinite positive, or negative, homogeneous dps $(1, x, \ldots)$,*
- *there exists a quadratic form $f$ such that for all primitive elements $x$ which are $\pm 1$ graded there exists an infinite positive homogeneous dps $(1, x, f(x), \ldots)$,*

*then the sequence groups of positive and negative homogeneous divided power series form a sequence pair.*

*Proof.* First, we investigate whether the restriction on the primitives is compatible with scalar extensions. Specifically, we need to check whether $H \otimes K$ for $K \in \Phi$-**alg** satisfies the conditions of the theorem as well. This is the case since the primitive elements are the kernel $U$ of the map $x \mapsto \Delta(x) - x \otimes 1 - 1 \otimes x$, and the kernel of this map in $H \otimes K$ must, therefore, be $U \otimes K$.

We show that if $1/2 \in \Phi$ and if there exists a dps over each $\pm 1$-graded primitive element, then there exists a quadratic form $f$ so that the second option of the either-statement is satisfied. Specifically, we know that $u = x(-x) = (1, 0, 2x_2 - x_1^2, \ldots) \in H^1_\pm$ for all $x \in G_\pm$. We get that $x(-1/2 \cdot_H u) = (1, x_1, x_1^2/2, \ldots) \in G_\pm$. Clearly, $x \mapsto x^2/2$ is a quadratic form. Note that $(x, s) \mapsto (1, x, f(x), \ldots) \times (1, 0, s, \ldots)$, going from the positively or negatively graded primitive elements, parametrizes all positive (or negative) homogeneous dps'es, since any other positive dps cannot differ in the first coordinate, nor can they differ in the second. Therefore, they must, by Lemma (2.4.22), form a sequence $\Phi$-group. To see that this construction is fully compatible with scalar extensions, realize that the groups $G_\pm(K)$ are generated, if one allows $K$-scalar multiplication, by the $G_\pm(\Phi)$.

It only remains to check whether the restrictions of Definition (2.4.2) are satisfied. Take $x \in G_\sigma(K)$ and $y \in G_{-\sigma}(K)$. We apply Corollary (2.4.19) to see that all the elements $[a, b]$ such that $[a, b] \neq [n, n]$ for all $n$, are part of positive or negative homogeneous dps'es. As such, we see that the operators $Q, T$ are defined. We also know that all $[a, b]$ with $3 \neq a/b > 2$ are zero, since the first non-zero $[qa, qb]$, for $0 < q \in \mathbb{Q}$ is necessarily primitive. But $[3, 1]$ is the only 2-graded $[a, b]$ with $a > 2b$. From Lemma (2.4.6), we conclude that conditions (2.5) and (2.6) hold. ■

# 3 Interlude: Special sequence pairs

In this chapter, we present some examples of sequence pairs. Specifically, we construct some sequence pairs over rings $\Phi$, with $1/6$ not necessarily in $\Phi$. We will also look at the easiest class of structurable algebras, namely the associative algebras with involution. We do this by studying a generalization of the broader class of 3-special Kantor pairs, as introduced by Allison and Faulkner [AF99, Section 6], and their generalizations to sequence pairs.

Let $A$ be a unital associative algebra over $\Phi$ with three orthogonal idempotents $e_{-1}, e_0, e_1$ which sum to 1. There is a 5-grading on $A$ depending on those idempotents. Namely, consider the $\mathbb{Z}$-grading defined by $A_i = \sum_{j-k=i} e_j A e_k$. We can look at the sequence group with elements in $(1, A_1, A_2)$ and the one with elements in $(1, A_{-1}, A_{-2})$. We denote the former, in this chapter, as $G_+$ and the latter as $G_-$. Note that the grading readily gives $(G_+, G_-)$ the structure of a sequence pair.

**Definition 3.1.1.** A sequence pair $P$ is **special** if there exists $(A, E, \theta)$, where $A$ is a unital associative algebra, $E$ a set of three orthogonal idempotents in $A$ and $\theta$ a defining sequence pair representation such that $\theta_+(P_+) \leq G_+$ and $\theta_-(P_-) \leq G_-$. If this is only a defining representation of a weak sequence pair $P$, then we call $P$ **weakly special**

*Remark* 3.1.2. Note that this is a generalization of special Jordan pairs. Our generalization is a slight adaptation of a generalization, namely 3-special Kantor pairs, of the special Jordan pairs.

**Proposition 3.1.3.** *If $1/2 \in \Phi$ and $P$ is a pair of sequence groups over $\Phi$, then $P$ is weakly special if and only if it is special.*

*Proof.* Note that special implies weakly special. So, we prove the converse. Specifically, we need to prove that $Q_x(y) = (1, [2,1], [4,2]) \in G_+(K)$ for $x \in G_+(K), y \in G_-(K)$. An easy computation shows that $[4,2] = (4,2) - [1,1][3,1]$. We assume that the representations are in standard form, i.e. $(x, 0) \mapsto (1, x_1, x_1^2/2)$. Suppose that $y = (y_1, s)$. We get

$$\mathrm{ad}_x^{(4)}(y_2) = \mathrm{ad}_x^{(4)}(y_1^2/2 + s) = Q_x^2(s)_2 + \sum_{i+j=4, ij\neq 0} [i,1][j,1]/2,$$

by the moreover part of Lemma (2.1.9). So, we see that

$$[4,2] = Q_x^2(s)_2 + \frac{1}{2} Q_x^1(y_1)_1^2 - \frac{1}{2}[[1,1], T_x(y)_2].$$

This means that $Q_x(y) \in G_+(K)$ if $\frac{-1}{2}[[1,1], T_x(y)_2] \in H_+^1(K)$. We know that $[1,1] = [x_1, y_1]$. So, we can express $[T_x(y), [1,1]]$ using the $(2,1)$ linearization of $T$, it is namely

$$T^{2,1}_{T_x(y),x}(y)_2 = \mathrm{ad}^{(2)}_{T_x(y)_2} \mathrm{ad}^{(1)}_x(y_1) = (\mathrm{ad}\, T_x(y)_2)(\mathrm{ad}\, x)(y_1) = [T_x(y)_2, [1,1]].$$

We note that all linearizations of $T$ map to $H^1(K)$. So, $P$ is special. ■

*Remark* 3.1.4. Suppose that $x, y, y_1, s$ are as in the previous proposition. If we write $Q_x(y_1, s) = (Q_x^1 y_1, Q'_x y + Q_x^2 s)$, then we see, if $1/2 \in \Phi$ and if both sequence groups are in standard form, that

$$Q'_x y = -\frac{-1}{2}[[1,1], T_x(y)],$$

so the question is whether such a $Q'$ exists for a general representation.

*Remark* 3.1.5. If $1/2 \notin \Phi$ it is not at all obvious how one could show that $(1, [2,1], [4,2])$ is an element of the group as it will depend on the quadratic form $x_1 \mapsto f(x_1)$ corresponding to the representation by $(x, 0) \mapsto (1, x_1, f(x_1))$ of $G_\sigma(K)$. The exact condition is

$$(4,2) - [1,1][3,1] - f([2,1]) = [4,2] - f([2,1]) \in H_2^1(K).$$

There is a nice correspondence between associative unital algebras with three idempotents, and such algebras which do have a $3 \times 3$-matrix form, although there may be entries of these matrices which lie in different modules. So, suppose that we have a unital associative algebra $A$ with three idempotents $e_{-1}, e_0, e_1$, then we can write each element $x$ of $A$ uniquely as a matrix

$$\begin{pmatrix} e_1 x e_1 & e_1 x e_0 & e_1 x e_{-1} \\ e_0 x e_1 & e_0 x e_0 & e_0 x e_{-1} \\ e_{-1} x e_1 & e_{-1} x e_0 & e_{-1} x e_{-1} \end{pmatrix},$$

and the multiplication of these matrices, which can be seen to be a subset of the $3 \times 3$ matrices over $A$, corresponds to the multiplication in $A$. If, on the other hand, we have an algebra of $3 \times 3$ matrices, then we can just take the three diagonal idempotents, i.e. the matrices which are zero everywhere except for a 1 on a single diagonal position.

*Example* 3.1.6 *(Hermitian special sequence pairs, if* $1/2 \in \Phi$*).* An important example of a class 3-special Kantor pairs, due to Allison and Faulkner [AF99, Section 8], and, thus, of a class of special sequence pairs, are the Kantor pairs from hermitian forms. Suppose $\mathcal{D}$ is an associative unital algebra with involution $x \longmapsto \bar{x}$ and let $\mathcal{X}$ be a left $\mathcal{D}$-module with a hermitian form $h : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{D}$, i.e. $h(d \cdot x, y) = dh(x, y)$ and $\overline{h(x, y)} = h(y, x)$, for $d \in \mathcal{D}$ and $x, y \in \mathcal{X}$. First, we construct the associative algebra in which it is special, and then we will construct the full sequence groups. Consider the matrices of the form

$$\begin{pmatrix} \mathcal{D} & \mathcal{X} & \mathcal{D} \\ \bar{\mathcal{X}} & \mathcal{E} & \bar{\mathcal{X}} \\ \mathcal{D} & \mathcal{X} & \mathcal{D} \end{pmatrix},$$

where $\mathcal{E}$ denotes

$$\{(\phi, \psi) \in \mathrm{End}_{\mathcal{D}}(\mathcal{X}) \oplus \mathrm{End}_{\mathcal{D}}(\mathcal{X})^{op} | h(\phi(x), y) = h(x, \psi(y)) \text{ for all } x, y \in \mathcal{X}\},$$

and where $\bar{\mathcal{X}}$ is just another copy of $\mathcal{X}$. However, $\bar{\mathcal{X}}$ will enjoy different actions of $\mathcal{D}$ and $\mathcal{E}$. We still need to define the multiplications, $\bar{\mathcal{X}}$ is a right $\mathcal{D}$-module under $\bar{x} \cdot d = \overline{\bar{d}x}$, a right $\mathcal{E}$-module under $x \cdot (\phi, \psi) = \psi(x)$ and a left $\mathcal{E}$-module under $(\phi, \psi)\bar{x} = \overline{\phi(x)}$. We still need to define left and right multiplications between elements of $\mathcal{X}$ and $\bar{\mathcal{X}}$. To accomplish that, we put $x\bar{y} = h(x, y)$ and $\bar{x}y = (A_{y,x}, A_{x,y}) \in \mathcal{E}$, where $A_{x,y}z$ equals $h(z, x)y$. This algebra is associative.

Now, we suppose that $1/2 \in \Phi$. We consider the group with elements

$$\begin{pmatrix} 1 & x & -h(x,x)/2 + s \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix},$$

where $x \in \mathcal{X}$ and $s$ is a skew element of $\mathcal{D}$, i.e. it is an element such that $\bar{s} = -s$. If we denote this matrix with $(x, s)$, one sees that $(a, b)(c, d) = (a + c, b + d + \frac{h(a,c) - h(c,a)}{2})$. This is easily seen to be a sequence group, by taking the sequence of grading components. We also consider the group of elements of the form

$$\begin{pmatrix} 1 & & \\ -\bar{y} & 1 & \\ -h(y,y)/2 + t & y & 1 \end{pmatrix},$$

where $y \in \mathcal{X}$ and $t$ a skew element. The only thing left to prove to show that this is a sequence pair, is to prove that the maps $Q^1, T, Q^2$ map to the right space. So, we compute

$$\begin{aligned}
\mathrm{ad}^{(2)}_{(x,s)} \begin{pmatrix} & & \\ -\bar{y} & & \\ & y & \end{pmatrix} &= \begin{pmatrix} (-h(x,x)/2 + s)y + h(x,y)x & & \\ & & \overline{(-h(x,x)/2 + s)y + h(x,y)x} \\ & & \end{pmatrix} \\
\mathrm{ad}^{(3)}_{(x,s)} \begin{pmatrix} & & \\ -\bar{y} & & \\ & y & \end{pmatrix} &= \begin{pmatrix} & & -h(x,x)h(y,x)/2 + sh(y,x) + h(x,y)h(x,x)/2 + h(x,y)s \\ & & \\ & & \end{pmatrix} \\
\mathrm{ad}^{(4)}_{(x,s)} \begin{pmatrix} & & \\ & & \\ t & & \end{pmatrix} &= \begin{pmatrix} & & (-h(x,x)/2 + s)t(-h(x,x) - s) \\ & & \\ & & \end{pmatrix}
\end{aligned}$$

These all lie in the right space. Hence, we have got a sequence pair. Notice that if you linearize $Q^1$, then you get the usual $V$. Specifically, one gets $(Q^1(z \cdot x) - Q^1_z - Q^1_x)(y) = h(x, y)z + h(z, y)x - h(z, x)y = V_{x,y}z$.

*Remark* 3.1.7. We note that if we linearize $Q^1$ in later sections, then we get $(Q^1(z \cdot x) - Q^1(z) - Q^1(x))(y) = -V_{x,y}z$. The reason for that, is that Allison and Faulkner [All79] use $-[xyz] = V_{x,y}z$ for the associated Lie triple system of a Kantor pair, while we used $[xyz] = V_{x,y}z$. In terms of the associated Lie triple system, one should always get that the linearization of $Q^1$ is $[z, [x, y]] = -[xyz]$.

We can generalize the previous example, so that there is no need for $1/2$. Namely, we chose a quadratic form $-h(x, x)/2$ which makes use of $1/2$. We could, all the same, work with any quadratic form $f$ which polarizes to

$$-h(x, y) + \psi(x, y)_\sigma,$$

for any bilinear form $\psi : \mathcal{X} \times \mathcal{X} \longrightarrow S$ with $S \leq \mathcal{D}$ the image of $x \mapsto x - \bar{x}$ (the skew elements, at least if $1/2 \in \Phi$) such that $\psi(x, y) - \psi(y, x) = h(x, y) - h(y, x)$ (note that this is no restriction, as $-h(x, y) + \psi(x, y)_\sigma$ should be symmetric). However, there is still one requirement on $f$, namely that $f(x) + \bar{f}(x) = -h(x, x)$. This is a necessary and sufficient condition for $Q^1_x(y)_1 \in (G_\sigma(K))_1$. If $1/2 \in \Phi$, then this additional condition is definitely satisfied, as its polarization is satisfied. When we have such a quadratic form, then we can look at

$$\theta_+(a, b) = \begin{pmatrix} 1 & x & f(x) + s \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix},$$

and the similar representation $\theta_-$ for the groups, with operation

$$(a, b)(c, d) = (a + c, b + d - \psi(a, c)).$$

However, the existence of such a quadratic form $f$ is not obvious at all. Notice that we only know that these are weakly special sequence pairs.

Note that associative algebras with involution are hermitian special sequence pairs, if we equip them with the hermitian form $(x, y) \mapsto x\bar{y}$. So, we give a specific example of a class of associative algebras without $1/2$, such that there exists such a quadratic form $f$. Since our class will not only envelop fields with characteristic 2 but also $\mathbb{Z}$-algebras, there are even fewer possible quadratic forms. Thus, the number of possibilities is quite small.

*Example* 3.1.8. Consider $A = R[x]/(x^2 - x - \alpha)$, for any commutative unital ring $R$ with $\alpha \in R$. We endow $A$ with an involution $a + bx \mapsto a + b - bx$, which means that $\bar{x} = 1 - x$ and $x\bar{x} = -\alpha$. Note that every separable field extension of degree 2 with Galois involution is such an algebra.

First, we compute $S$:

$$a + bx - \overline{a + bx} = bx - b(1 - x) = (2x - 1)b,$$

so $S = R(2x - 1)$. We also compute

$$h(a + bx, c + dx) = (a + bx)\overline{(c + dx)} = ac + ad - adx + bcx - bd\alpha.$$

We set

$$\psi(a + bx, c + dx) = (1 - 2x)(ac + ad - \alpha bd),$$

which means that

$$h(a + bx, c + dx) - \psi(a + bx, c + dx) = 2xac + xad + xbc - 2x\alpha bd.$$

So, we set $f(a + bx) = xa^2 + xab - x\alpha b^2 = x(a + bx)\overline{(a + bx)} = xN(a + bx)$, with $N$ the norm associated with $A$ and the involution. It is obvious that $f(a + bx) + \overline{f(a + bx)} = N(a + bx)$.

This means that $A \times S$ gives rise to a weakly special pair of sequence groups, whereby we need to look at $-f$ instead of $f$.

**Proposition 3.1.9.** *For each $A = R[x]/(x^2 - x - \alpha)$ where $R$ is a commutative unital ring, $\alpha \in R$ and involtution on $A$ defined by $x \longmapsto 1 - x$, there exists a special sequence pair $(A \oplus S, A \oplus S)$ with defining representation in*

$$\begin{pmatrix} A & A & A \\ A^{op} & \mathcal{E} & A^{op} \\ A & A & A \end{pmatrix}.$$

*Proof.* We already know that

$$(a, s) \longrightarrow \begin{pmatrix} 1 & a & -N(a)x + s \\ & 1 & -\bar{a} \\ & & 1 \end{pmatrix},$$

and

$$(a, s) \longrightarrow \begin{pmatrix} 1 & & \\ -\bar{a} & 1 & \\ -N(a)x + s & a & 1 \end{pmatrix},$$

form a weakly special sequence pair. We want to show that $[4, 2] + N([2, 1])x \in R(1 - 2x)$. By Proposition (3.1.3), we know that the proposition holds if $1/2 \in R$. Suppose now that $R$ has no 2-torsion. We note that $k(1 - 2x) \in R(1 - 2x) \otimes R[1/2]$, and $k(1 - 2x) \in A = R[x]/(x^2 - x - \alpha)$ imply that $k(1 - 2x) \in R(1 - 2x)$, as $k(1 - 2x) \in A$ implies that $k \cdot 1 \in R \subset A$. Now, we see that the proposition holds for general $R$, as each commutative ring is a quotient of a commutative ring without 2-torsion. ∎

*Remark* 3.1.10. We consider a family $\mathcal{F}$ of unital associative $\mathbb{Z}$-algebras with involution. We set $S_A = \{x - \bar{x} | x \in A\}$ for each $A$ in $\mathcal{F}$, and suppose that we have quadratic forms $f$ such that $f(x, y) - x\bar{y} \in S_A$ for all $x, y \in A$. Suppose that $A \in \mathcal{F}$ implies that $A \otimes K \in \mathcal{F}$ for all unital commutative associative $\mathbb{Z}$-algebras $K$ and that this operation is compatible with the quadratic form $f$ and the involution. If each $A \in \mathcal{F}$ is of the form $A' \otimes K$ for some $A'$ without 2-torsion and if $S_{A\otimes K} \cap A \otimes 1 = S_A \otimes 1$ for all $A$, then we can generalize the previous proposition to this family. We call such $\mathcal{F}$ a **well behaved associative family**.

Now, we look at algebras which behave like quaternion algebras.

*Example* 3.1.11. Consider

$$B = \left\{ \begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} : x, y \in A \right\},$$

with $A = R[x]/(x^2 - x - \alpha)$ as before, and $\beta \in R$. We define an involution

$$\begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} & -\beta\bar{y} \\ -y & x \end{pmatrix}.$$

We determine what $S$ should be, by computing

$$\begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} - \begin{pmatrix} \bar{x} & -\beta\bar{y} \\ -y & x \end{pmatrix} = \begin{pmatrix} x - \bar{x} & 2\beta\bar{y} \\ 2y & \bar{x} - x \end{pmatrix}.$$

So, we see that

$$S = \left\{ \begin{pmatrix} s & 2\beta\bar{y} \\ 2y & -s \end{pmatrix} : s \in R(1 - 2x), y \in R \right\}.$$

We also compute

$$h((x, y), (a, b)) = \begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} \bar{a} & -\beta\bar{b} \\ -b & a \end{pmatrix} = \begin{pmatrix} \bar{a}x - \beta b\bar{y} & -\beta x\bar{b} + \beta a\bar{y} \\ \bar{a}y - b\bar{x} & -\beta y\bar{b} + a\bar{x} \end{pmatrix}.$$

Hence, we get

$$h((x, y), (a, b)) - h((a, b), (x, y)) = \begin{pmatrix} x\bar{a} - a\bar{x} + \beta(y\bar{b} - b\bar{y}) & 2\beta(-x\bar{b} + a\bar{y}) \\ 2(y\bar{a} - b\bar{x}) & -x\bar{a} + a\bar{x} - \beta(y\bar{b} - b\bar{y}) \end{pmatrix}.$$

So, we can think of it as

$$(x, y)\overline{(a, b)} - (a, b)\overline{(x, y)} = (x\bar{a} - a\bar{x} + \beta(y\bar{b} - b\bar{y}), 2(y\bar{a} - b\bar{x})).$$

So, if we use the same $\psi$ as in the previous example, to define

$$\psi'((x, y), (a, b)) = (\psi(x, a) - \beta\psi(b, y), -2b\bar{x}),$$

then we get

$$h((x, y), (a, b)) - \psi'((x, y), (a, b)) = (f(x, a) - \beta f(b, y), \bar{a}y + b\bar{x}),$$

with $f(x, a)$ the polarization of the quadratic form of the previous example. The previous expression is a polarization of the quadratic form

$$f'(x, y) = (f(x) - \beta f(y), y\bar{x}).$$

If we rewrite this, using the usual norm $N$ on the quaternion algebras, we get

$$f'(a, b) = (N(a, b)x, -b\bar{a}).$$

This $f'$ satisfies $(f' + \bar{f}')(x_1) = N(x_1)$.

*Remark* 3.1.12. The previous example is not a well behaved associative family, since $S \otimes \mathbb{Z}[1/2] \cap B$ is not equal to $S$ for algebras $B$ which have no 2-torsion and $1/2 \notin B$. The exact problem is, is that there could be elements $x, y$ such that

$$[4,2] - Q([2,1]) = \begin{pmatrix} 0 & \beta\bar{y} \\ y & -0 \end{pmatrix} \mod S,$$

for $y$ not divisible by 2, since the diagonal part $D$ of $S$ satisfies

$$D_{A\otimes K} \cap A = D_A \otimes .1$$

*Example* 3.1.13. We consider, again,

$$B = \left\{ \begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} : x, y \in A \right\},$$

but now with a wrong involution, which does not make it into a composition algebra. We define the involution[1] as

$$\begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} & \beta\bar{y} \\ y & x \end{pmatrix}.$$

Now, we see that

$$S = \left\{ \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} : s \in R(1-2x) \right\}.$$

This new $S$ has, clearly, better properties than the previous $S$. One easily checks that

$$f'(a,b) = (N(a)x + \beta N(b)x, b\bar{a}),$$

is a quadratic form satisfying all necessary properties. We note that these algebras form a well behaved associative family and give, therefore, all rise to special sequence pairs.

**Proposition 3.1.14.** *Suppose $B$ is an algebra of the form*

$$B = \left\{ \begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} : x, y \in A \right\},$$

*for $A = R[x]/(x^2 - x - \alpha)$, with involution*

$$\begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} & -\beta\bar{y} \\ -y & x \end{pmatrix},$$

*then $B$ induces a special sequence pair with defining representation in*

$$\begin{pmatrix} B & B & B \\ B^{op} & \mathcal{E} & B^{op} \\ B & B & B \end{pmatrix}.$$

*Proof.* We know that the proposition holds if $1/2 \in B$ or if everything in $B$ is 2-torsion. So, we know for each $B$ that either $1/2 \in B$ and the propositions holds, or $B \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong B/(2)$ satisfies the proposition. So, suppose $[4,2] - Q([2,1]) \notin S$ for some $x \in G_{\pm}(K), y \in G_{\mp}(K)$. We know that the only possible problem, see Remark (3.1.12), is that this element could be of the form

$$\begin{pmatrix} 0 & \beta\bar{y} \\ y & 0 \end{pmatrix} \mod S,$$

[1] This involution is not random. It is not the usual involution corresponding to the Cayley-Dickinsonproces, but is the involution of that same construction interpreted as the construction of a hermitian structurable algebra.

for $y$ not divisible by 2. We know that $([4,2] - Q([2,1])) \otimes 1 \in S_{\mathbb{Z}/2\mathbb{Z}}$. Now, we note that the theorem follows, because of the form of the $S$, namely in both cases we know that

$$\operatorname{nondiag}(S) = \left\{ \begin{pmatrix} 0 & 2\beta\bar{y} \\ 2y & 0 \end{pmatrix} : y \in R \right\},$$

with nondiag the projection

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

■

*Remark* 3.1.15. This construction will not work for each family of associative algebras with involution. Consider, for example, $R[x]/(x^2 - 1)$ with involution $x \mapsto -x$. This algebra behaves very differently depending on the existence of 2-torsion.

# 4 The universal representation

In this chapter, we construct the universal (sequence pair) representation of a sequence pair. This will be a $\mathbb{Z}$-graded Hopf algebra. Thereafter, we investigate the (Jordan-)Kantor-like sequence pairs. These are sequence pairs with some additional assumptions. Lastly, we consider sequence pairs defined from hermitian structurable algebras.

## 4.1 The universal representation

Now, we prove some lemmas and a theorem generalizing [Fau00, Lemma11-14, Theorem 15]. Most of the lemmas correspond to the existence of operators $\Delta$, $S$ and how they interact. This will be used to show that the universal representation together with these operators and some others forms a Hopf algebra.

To avoid the need for duplication in what follows, we formulate the lemmas and the theorem as general as possible. So, we will consider in most lemmas a pair of sequence $\Phi$-groups $(G_+, G_-)$ with a pair of sequence group representations in the same algebra $A$. We will call such representations **paired representations**. For these, we can consider the elements $(a, b)$ and $[a, b]$ for $a, b \in \mathbb{N}$, defined from $x \in G_\sigma(K), y \in G_{-\sigma}(K)$. We interleave the lemmas with the immediate variants for sequence pairs as corollaries, to give the reader some understanding of what we try to achieve.

**Lemma 4.1.1.** *If $\rho : G \longrightarrow A$ is a paired representation and if $\phi : A \longrightarrow B$ is an algebra morphism, then $\phi \circ \rho$ is a paired representation. Moreover, $\phi(a, b) = (a, b)$ and $\phi[a, b] = [a, b]$ hold for all $a, b \in \mathbb{N}$ and $x \in G_\sigma(K), y \in G_{-\sigma}(K)$.*

*Proof.* Trivial. ∎

**Corollary 4.1.2.** *If $\rho : G \longrightarrow A$ is a sequence pair representation and if $\phi : A \longrightarrow B$ is an algebra morphism, then $\phi \circ \rho$ is a sequence pair representation.*

**Lemma 4.1.3.** *Suppose $\rho, \xi : G \longrightarrow A, B$ are paired representations. Then*

$$(\rho \otimes \xi)^\sigma_n : g \longmapsto \sum_{k+l=n} \rho^\sigma_k(g) \otimes \xi^\sigma_l(g)$$

*is paired representation of $G$ in $A \otimes B$. Moreover, if $x \in G_\sigma(K), y \in G_{-\sigma}(K)$, then for all $a, b$ coprime, we have $[na, nb] = \sum_{i+j=n} [ia, ib] \otimes [ja, jb]$ for $n \in \mathbb{N}$.*

*Proof.* We denote $(\rho \otimes \xi)^\sigma_n(x)$ as $x^n$ and $\rho_n(x), \xi_n(x)$ as $x_n$ (these symbols will be unambiguous). We are working over a general $K$, so that we get the result without really interacting with extensions of scalars. We will prove the 3 properties of Proposition (2.1.5), to demonstrate that it is a pair of sequence group representations. This representation is clearly even, as $n$ is odd implies that it can only be written as a sum of an odd with an even integer, hence it inherits evenness from $A$ and $B$.

The first and third property of the proposition are obvious. So, we only need to show that $\rho \otimes \xi$ is a group homomorphism to prove that it is is a pair of sequence group representations:

$$\begin{aligned}\sum_{i+j=n} x^i y^j &= \sum_{p+q+r+s=n} (x_p \otimes x_q)(y_r \otimes y_s) \\ &= \sum_{p+q+r+s=n} x_p y_r \otimes x_q y_s \\ &= \sum_{e+f=n} (xy)_e \otimes (xy)_f \\ &= (xy)^n.\end{aligned}$$

We also compute that

$$\begin{aligned}\mathrm{ad}_x^{(n)}(a \otimes b) &= \sum_{i+j=n} x^i a \otimes b(x^{-1})^j \\ &= \sum_{p+q+r+s=n} x_p a x_q^{-1} \otimes x_r b x_s^{-1} \\ &= \sum_{e+f=n} \mathrm{ad}_x^{(e)}(a) \otimes \mathrm{ad}_x^{(f)}(b). \end{aligned} \tag{4.1}$$

We can conclude that

$$\mathrm{ad}_x^{(n)}(y^m) = \sum_{k+l=n,p+q=m} \mathrm{ad}_x^{(k)} y_p \otimes \mathrm{ad}_x^{(l)} y_q, \tag{4.2}$$

since $y^m = \sum_{i+j=m} y_i \otimes y_j$.

Now, we need to determine some properties the elements $[n,m]$, $(n,m)$ for $n,m \in \mathbb{N}$. We want to apply Theorem (2.3.3). Equation (4.2) shows that

$$(n,m) = \mathrm{ad}_x^{(n)}(y_m) = \sum_{i+j=n,k+l=m} (i,k) \otimes (k,l).$$

So, we can apply Theorem (2.3.3) with as relation $\Delta$ the equality (where we interpret the lefthandside to be in the algebra $A \otimes B$ and the right hand side to be a tensor product of algebras), to prove that

$$[an, am] = \sum_{i+j=a} [in, im] \otimes [jn, jm], \tag{4.3}$$

if $n$ and $m$ are coprime. ■

**Corollary 4.1.4.** *Suppose that $G$ is a sequence pair. If $\rho$ is a sequence pair representation of $G$ in $A$ and if $\xi$ is a sequence pair representation of $G$ in $B$. Then*

$$(\rho \otimes \xi)_n^\sigma : g \longmapsto \sum_{k+l=n} \rho_k^\sigma(g) \otimes \xi_l^\sigma(g)$$

*is a sequence pair representation of $G$ in $A \otimes B$.*

*Proof.* From Lemma (4.1.3) we immediately get that $Q$ and $T$ exist and coincide with the usual $Q$ and $T$. Moreover, restrictions (2.5) and (2.6) are immediately satisfied as well, using Lemma (2.4.6) to consider the equivalent restriction $[a,b] = 0$ for $3 \neq a/b > 2$. ■

**Lemma 4.1.5.** *Suppose that $\rho : G \longrightarrow A$ is a paired representation, then*

$$\rho \circ (\cdot^{-1}) : G \longrightarrow A^{op}$$

*is a paired representation. Moreover, the elements $(a,b)_{x,y}$ and $[a,b]_{x,y}$ computed for $x \in G_{\pm}(K)$, $y \in G_{\mp}(K)$, using $\rho$ are related to the similar elements $(a,b)'_{x,y}$, $[a,b]'_{x,y}$ computed using $\rho \circ (\cdot^{-1})$ by*

$$(a,b)_{x,y} = (a,b)'_{x,y^{-1}},$$
$$\exp(t[a,b]_{x,y}) \exp(t[a,b]'_{x,y}) = 1.$$

*Proof.* $\rho_\sigma$ is a group homomorphism from $G_\sigma(K)$ to $D_K \subset (A \otimes K)^{\mathbb{N}}$, which is in itself a sequence $\Phi$-group. It is obvious that $D^{op}$ can be seen as a sequence $\Phi$-group in $A^{op}$. But $\rho_\sigma \circ (^{-1})$ is a $\Phi$-group homomorphism from $G_\sigma \longrightarrow D^{op}$ and this morphism commutes with scalar multiplication. Hence, it is a sequence $\Phi$-group representation.

We note, for all $x \in G_\sigma(K), y \in G_{-\sigma}(K)$, that $\mathrm{ad}_x^{(n)}(y_n) = \sum_{i+j=n}(x^{-1})_i \cdot (y^{-1})_n \cdot x_j$ in $A^{op}$ for $\rho \circ (\cdot^{-1})$ (we used $(x^{-1})_1$ to denote the $x_i$ of $\rho \circ (\cdot^{-1})$ to avoid confusion). Evaluated in $A$, we get that it coincides with the $\mathrm{ad}_x^{(n)}((y^{-1})_n)$ from $\rho$. This proves the relation between the elements of the form $(a,b), (a,b)'$.

We note that $[a,b]$, in $A$, is uniquely determined by

$$\exp(ty)^{(\exp(sx)^{-1})} = \sum s^a t^b (a,b) = \prod_{a,b \text{ coprime}} \exp(s^a t^b [a,b]),$$

where the order of the product is increasing on the fractions $a/b$. We note that conjugating with $\exp(sx)^{-1}$ in the usual representation $\rho$, is the same as conjugating with $\exp(sx)^{-1}$ in the representation $\rho \circ (\cdot^{-1})$ in consideration (it is conjugation with the same element, instead of its inverse, since we take an inverse and then multiply in the opposite order). So, if we write the computations for $\rho \circ (\cdot^{-1})$ as we would do them in $A$ and in terms of the usual representation $\rho$, we get

$$\exp(ty^{-1})^{(\exp(sx))^{-1}} = \prod_{a,b \text{ coprime}} \exp(s^a t^b [a,b]'),$$

with the order of the product decreasing on the fractions $a/b$. The product of these expressions in $A$ is

$$1 = \exp(ty)^{(\exp(sx)^{-1})} \exp(ty^{-1})^{(\exp(sx)^{-1})} = \prod_{a,b \text{ coprime}} \exp(s^a t^b [a,b]) \prod_{a,b \text{ coprime}} \exp(s^a t^b [a,b]'),$$

with the first product increasing on $a/b$, and the second product decreasing on $a/b$. Now, we want to prove the moreover-part. Specifically, we use that

$$1 = \prod_{a,b \text{ coprime}} \exp(s^a t^b [a,b]) \prod_{a,b \text{ coprime}} \exp(s^a t^b [a,b]').$$

So, if we set $\exp(t[a,b]'') = \exp(t[a,b]')^{-1}$, then we have

$$\prod_{a,b \text{ coprime}} \exp(s^a t^b [a,b]'') = \prod_{a,b \text{ coprime}} \exp(s^a t^b [a,b]),$$

where the order of both products is increasing on $a/b$. So, we see that all $[a,b] = [a,b]''$ (even for $a,b$ not coprime), as they correspond to the unique elements $[a,b]$ defined from the $(a,b)$. ■

**Corollary 4.1.6.** *If $\rho : G \longrightarrow A$ is a sequence pair representation of $G$ in $A$, then $\rho \circ (\cdot^{-1})$ is a sequence pair representation of $G$ in $A^{op}$.*

*Proof.* Lemma (4.1.5) shows that this is the case. Specifically, the $Q$ and $T$ of $\rho \circ (\cdot^{-1})$ and $\rho$ correspond to each other and the restrictions (2.5), (2.6) are preserved (most clearly in their equivalent form of Lemma (2.4.6)). ■

Let $G$ be a pair of sequence $\Phi$-groups and let $P$ be a class of representations of $G$ so that $\rho, \xi \in S$ implies that $\phi \circ \rho, \rho \otimes \xi, \rho \circ (\cdot^{-1}) \in P$ for all algebra morphisms $\phi$ with a suitable domain. We also suppose that the trivial representation ($G \longrightarrow \Phi$, with sequence group representations $g \mapsto (1)$) is in $P$. A class of such representations of a pair $G$ are called $P$-**representations** and we call $P$ a **sensible collection of representations** of $G$.

**Corollary 4.1.7.** *Suppose $G$ is a sequence pair, then the sequence pair representations of $G$ form a sensible collection of representations.*

**Lemma 4.1.8.** *Let $P$ be a sensible collection of representations of $G$. Suppose that $\rho : G \longrightarrow A$ is a $P$-representation, then $\hat{\rho}$ with $\hat{\rho}^{\sigma}_n(x) = ad^{(n)}_x$ is a $P$-representation.*

*Proof.* Consider $m : A \otimes A^{op} \longrightarrow End_{\Phi}(A)$ defined by $m(a \otimes b)(c) = acb$ and let $\chi = \rho \circ (^{-1})$. Then we can write

$$m \circ (\rho \otimes \chi) = \hat{\rho}.$$

Hence, $\hat{\rho}$ is a $P$-representation of $G$ in $A$. ■

**Corollary 4.1.9.** *If $\rho : G \longrightarrow A$ is a sequence pair representation of $G$ in $A$, then $\hat{\rho}$, with $\hat{\rho}^{\sigma}_n(x) = ad^{(n)}_x$, is a sequence pair representation.*

*Remark* 4.1.10. The previous corollary shows, among other things, that we are justified to speak of the adjoint representation of sequence pairs.

**Definition 4.1.11.** For a sequence pair $G$, the **universal (sequence pair) representation** is a unital associative algebra $U$ together with a sequence pair representation $\phi$ such that for all sequence pair representations $\psi : G \longrightarrow A$, there exists a unique algebra morphism $\theta : U \longrightarrow A$, so that $\theta \circ \phi = \psi$.

Now we will construct the universal representation for sequence pairs. Take the unital associative algebra $B$ generated by symbols $g_i$ for $i \in \mathbb{N}$ and $g, g' \in G_{\pm}(\Phi)$. We take the quotient with respect to the following relations:

1. $g_0 = 1$,
2. $(\lambda \cdot g)_n = \lambda^n g_n$,
3. $(gg')_n = \sum_{i+j=n} g_i g'_j$,
4. $(1)_n = 0, n > 0$,
5. $h_i = 0$ for $h \in H^1_{\sigma}(\Phi), i$ odd,
6. $g_j g_i = \sum_{a+2b=i+j} \binom{a}{i-b} g_a (g_1^2 - 2g_2)_{2b}$,
7. $[g_j, g'_i] = \sum_{\substack{a+c=i \\ b+c=j \\ c \neq 0}} g'_a g_b [g, g']_{2c}$,

where the last two relations are equations (2.1) and (2.2). We call this quotient $B'$. We can give $B$ a $\mathbb{Z}$-grading by putting $g_n \in B_{\pm n}$ with the sign the same as the group $G_\pm$ of which $g$ is an element. Since the asked relations are compatible with this grading, we see that $B'$ inherits this grading. We have sequence $\Phi$-group representations of $G_+$ and $G_-$ by mapping

$$g \longmapsto (1, g_1, \ldots, g_n, \ldots),$$

for $g \in G_\pm(\Phi)$. Now, we divide out the restrictions that are required to make this a sequence pair. To be specific, we divide out by the ideal generated by

$$\begin{aligned} T_x(y)_n =& [3n, n], \\ Q_x(y)_n =& [2n, n], \\ [a, b] =& 0 \qquad\qquad \text{for } 3 \neq a/b > 2, \end{aligned}$$

and all the linearizations of these expressions for all $x \in G_\sigma(\Phi), y \in G_{-\sigma}(\Phi), n \in \mathbb{N}$. Note that we used the condition of Lemma (2.4.6) instead of conditions (2.5) and (2.6), as they are equivalent. All used relations are compatible with the grading. We call the algebra we have constructed $U(G)$ and we denote the sequence pair representation of $G$ in it with $\gamma$. This is, clearly, a universal sequence pair representation, as all relations must necessarily hold in any representation of $G$.

**Theorem 4.1.12.** *Suppose that $P$ is a sensible collection of representations of $G$ and suppose that $U(G)$ is a universal $P$-representation generated by the $g_i$, for $g \in G_\pm(\Phi)$ and $i \in \mathbb{N}$, then $U(G)$ is a cocommutative Hopf algebra. If, in addition, $U(G)$ has a $\mathbb{Z}$-grading as an algebra with $g_i$ being $\sigma i$ graded if $g \in G_\sigma(\Phi)$, then it is a $\mathbb{Z}$-graded Hopf algebra. Moreover, if there exists a faithful $P$-representation, then $\gamma$ is a faithful representation.*

*Proof.* We first prove the moreover-part. It is clear that if the first $i$ elements in $\gamma(g)$ are zero, then $g$ must be an element of $H^i(K)$, as there exists a faithful representation that factors through the universal representation. Since the representations are even, we also know that $g \in H^1(K)$ implies that $(g)_1 = 0$. Hence, the representation is faithful.

We know that $U = U(G)$ is a unital associative algebra, so we have maps $\mu : U \otimes U \longrightarrow U$ and $\eta : \Phi \longrightarrow U$ which correspond to the multiplication and unit. Since $P$ is sensible, we know that $\gamma \otimes \gamma$ is a $P$-representation of $G$. We define $\Delta : U \longrightarrow U \otimes U$ as the unique map such that $\Delta \circ \gamma = \gamma \otimes \gamma$. We note that

$$\Delta(g_n) = \sum_{i+j=n} g_i \otimes g_j.$$

As $P$ is sensible, we know that $\gamma \circ (\cdot^{-1})$ is a $P$-representation of $G$ in $U^{op}$. We take the unique $S : U \longrightarrow U^{op}$ such that $S \circ \gamma = \gamma \circ (\cdot^{-1})$. For the final map, the counit $\epsilon$, we use that there always exists a $P$-representation $\rho : G \longrightarrow \Phi$, namely the trivial representation.

Now, we show that $U$ forms, together with these maps, a Hopf algebra. We see that $\Delta$ is coassociative on the generators $g_i$ of $U$, and thus on the whole of $U$, as

$$(\mathrm{Id} \otimes \Delta) \circ \Delta(g_n) = \sum_{i+j+k=n} g_i \otimes g_j \otimes g_k = (\Delta \otimes \mathrm{Id}) \circ \Delta(g_n).$$

We also see that $\epsilon$ is the counit, since

$$(\epsilon \otimes \mathrm{Id}) \circ \Delta(g_n) = \sum_{i+j=n} \epsilon(g_i) g_j = \mathrm{Id}(g_n) = (\mathrm{Id} \otimes \epsilon) \circ \Delta(g_n).$$

Now, consider the algebra homomorphism $A : U \longrightarrow \mathrm{End}_\Phi(U)$ defined by

$$A = m \circ (\mathrm{Id} \otimes S) \circ \Delta \qquad \text{with } m(a \otimes c)(b) = abc.$$

On generators $g_n$, we see that $A(g_n) = \mathrm{ad}_g^{(n)}$. Note that

$$A(g_n)(1) = \mathrm{ad}_g^{(n)}(1) = \delta_{n0} = \epsilon(g_n)1 \qquad \text{with } \delta_{ij} \text{ the Kronecker delta.}$$

Since $A$ is an algebra homomorphism, we conclude that $A(x)(1) = \epsilon(x)1 = \eta \circ \epsilon(x)$ for all $x \in U$, which proves that

$$\mu \circ (\mathrm{Id} \otimes S) \circ \Delta = \eta \circ \epsilon.$$

Similarly, we can show that

$$\mu \circ (S \otimes \mathrm{Id}) \circ \Delta = \eta \circ \epsilon,$$

by considering $\mathrm{ad}_{g_{-1}}^{(n)}$. Hence, $U$ is a Hopf algebra. It is also cocommutative since it is cocommutative on the generators. Note that the maps $S, \Delta, \epsilon$ are compatible with the $\mathbb{Z}$-grading on $U$ as an algebra, hence it is also a Hopf grading. ■

**Corollary 4.1.13.** *The universal representation $(U(G), \gamma)$ of a sequence pair $G$ is a cocommutative $\mathbb{Z}$-graded Hopf algebra and $\gamma$ is a defining representation.*

*Remark* 4.1.14.

- One easily constructs the universal weak sequence pair representation by constructing the same algebra $B'$ in which both sequence $\Phi$-groups have representations and then dividing out by

$$\begin{aligned} T_x(y)_n &= [3n, n] && x \in G_\sigma(\Phi), y \in G_{-\sigma}(\Phi), \\ Q_x^2(h)_n &= [2n, n] && x \in G_\sigma(\Phi), h \in H^1_{-\sigma}(\Phi), \\ Q_x^1(y) &= [2, 1] && x \in G_\sigma(\Phi), y \in G_{-\sigma}(\Phi), \\ [a, b] &= 0 && a/b > 3, x \in G_\sigma(\Phi), y \in G_{-\sigma}(\Phi), \\ [a, b] &= 0 && a/b > 2, x \in G_\sigma(\Phi), h \in H^1_{-\sigma}(\Phi). \end{aligned}$$

  One easily sees, using Lemmas (4.1.1), (4.1.3) and (4.1.5), that being the weak sequence pair representations of a weak sequence pair form a sensible collection of representations. So, the universal weak sequence pair representation is a $\mathbb{Z}$-graded Hopf algebra.

- We did set up the structure of the lemmas and the theorem so that we can later easily introduce Jordan-Kantor-like sequence pairs without needing to repeat the arguments of this section. Specifically, if $1/2 \in \Phi$, these are sequence pairs with an additional operator $P$. Moreover, we will also be able to use this theorem to prove that the universal representation of an extended version of these Jordan-Kantor-like sequence pairs to general $\Phi$, is a Hopf algebra.

- Suppose that $x \in G_\pm(K), y \in G_\mp(K)$, for a sequence group $G$. By Lemma (2.4.18), we know that the family $(n, m) = \mathrm{ad}_x^{(n)}(y_m)$ satisfies the conditions of Theorem (2.3.3). Therefore, we see for $a, b$ coprime, that $(1, [a, b], [2a, 2b], \ldots)$ is a divided power series.

- We assumed that the representations were representations of sequence $\Phi$-groups. However, this restriction is not necessary. We only made that assumption because we are only interested in sequence $\Phi$-groups. A small part where the assumption of class 2 (although we formulated the restriction in fact for sequence $\Phi$-groups) plays a role, is the proof that the universal $P$-representation is faithful if there exists a faithful representation, making use of the fact that we assumed that all representations are even.

So, to introduce a good notion of a universal representation for pairs of sequence groups of class higher than 2, one needs to decide how one generalizes evenness. Although just requiring that all $P$-representations $\rho$ satisfy $\rho(H^n)_k = 0$ if $0 \neq k \leq n$, might just do the job (notice, however, that this would have different properties).

**Proposition 4.1.15.** *Suppose that $G$ is a sequence pair such that $H^1_\sigma = 0$ for $\sigma = \pm 1$, then $G$ forms a Jordan pair with the operator $Q$.*

*Proof.* This is an adaptation of a theorem of Faulkner [Fau00, Theorem 5]. Specifically, it is an adaptation of the part of the proof where the axioms of a Jordan pair are proved.

First, we note that $(x^{-1}) = -x$ for all $x \in G_\pm(K)$. We recall the map $A = m \circ (\mathrm{Id} \otimes S) \circ \Delta$ with $m(a \otimes b)(c) = acb$, from any Hopf algebra into its endomorphism algebra. Take $x, z \in G_\sigma(K), y, w \in G_{-\sigma}(K)$. We note that $A_{x_i} = \mathrm{ad}^{(i)}_x$ for any $x \in G_\sigma(K)$. Now, we determine the operator $D_{x,y}(z)_1 = Q_{x,z}(y)_1 = \mathrm{ad}^{(1)}_x \mathrm{ad}^{(1)}_z(y_1) = -A_{x_1}A_{y_1}(z_1) = -[x,[y,z]] = -[[x,y],z]$.

We know that $A_{x_3}(y_1) = T_x(y)_2 = 0$. So, if we apply $A$ to both sides, we get

$$A_{x_3}A_{y_1} - A_{y_1}A_{x_3} - A_{x_2}A_{y_1}A_{x_1} + A_{x_1}A_{y_1}A_{x_2} = 0.$$

If we let both sides act on $w$, we get, since $[y_1, w_1] \in H^1_{-\sigma}(\Phi) = 0$ and $T_x(w) \in H^1_\sigma(\Phi) = 0$ that

$$Q_x D_{y,x} - D_{x,y} Q_x = 0.$$

This is the first axiom for Jordan pairs.

The second axiom is easily checked using $2y_2 = y_1^2$ to verify

$$[Q_x(y)_1, y_1] = [x_1, Q_y(x)_1].$$

The last axiom follows from

$$(Q_x(y))_2 = A_{x_4}(y_2) = x_4y_2 - x_3y_2x_1 + x_2y_2x_2 - x_1y_2x_3 + y_2x_4,$$

and letting it act on $w$. This means that

$$Q_{Q_x(y)} = A_{x_4}A_{y_2} - A_{x_3}A_{y_2}A_{x_1} + Q_xQ_yQ_x = Q_xQ_yQ_x,$$

since $A_{y_2}(w_1)$ can be computed from $(tw)^{sy}_3 = (tw)_3$ as the term belonging to $s^2t$, so it is zero. Similarly, we get that

$$A_{y_2}[x_1, w_1] = [Q_y(x)_1, w_1] + [[y_1, x_1], 0] + [x_1, A_{y_2}w_1] = 0,$$

where the interaction with the Lie bracket is a consequence of Lemma (2.1.9). ∎

## 4.2 Kantor-like sequence pairs

In this section, we investigate a specific subset of the (weak) sequence pairs that share certain properties with representations corresponding to sequence groups associated with Kantor pairs. To be exact, we are interested in three properties, which we will already mention, but not formalize yet. So, suppose that $G$ is a sequence pair. Firstly, we need that $H^1_\pm = [G_\pm, G_\pm]$. Secondly, we want that the $H^1_\pm$ have a faithful action (we will spell out in this section what this exactly means). Thirdly, we want the possibility to add a grading element in the TKK Lie algebra so that there is still a representation in the endomorphism algebra of this Lie algebra. Note that we will also need to generalize the TKK Lie algebra in this section, in order to formalize the third property. We will, in fact, see that it is always possible to add a grading element.

Now, we spell out what it means for $H^1_\pm$ to have a faithful action. We have maps

$$Q^1 : H^1(K) \longrightarrow \mathrm{Hom}(G_\mp(K), G_\pm(K)).$$

We want that if

$$Q^1(h)(xH^1_\mp(\Phi))H^1_\pm(\Phi) = Q^1(g)(xH^1_\mp(\Phi))H^1_\pm(\Phi)$$

holds for all $x \in G_\mp(\Phi)$, then $h = g$. This formulation may seem odd. Another formulation, which makes use of the universal representation (or any other defining representation), would be that if $h, g \in H^1_\pm(\Phi)$ and if $[h_2, x_1]$ equals $[g_2, x_1]$ for all $x \in G_\mp(\Phi)$, then $h$ and $g$ should be equal.

**Definition 4.2.1.** We call a (weak) sequence pair $G$ with $H^1_\pm = [G_\pm, G_\pm]$ and faithful actions of $H^1_\pm$ in the just described sense, a **Kantor-like (weak) sequence pair**.

*Remark* 4.2.2. These properties hold for the sequence pairs constructed from Kantor pairs in Theorem (2.4.8), by Proposition (1.11.3), as the spaces $L_{2\sigma}$ in the TKK Lie algebra, consist exactly out of the operators $K(x,z)$ with $K(x,z)(y) = V_{x,y}(z) - V_{z,y}(x)$, and $[x,z] = K(x,z)$ in the Lie algebra. Moreover, the equalities between those operators are exactly determined by their action on $L_{-\sigma}$. In the broader context of sequence groups related to Jordan-Kantor pairs, the Kantor pairs are exactly those pairs for which the sequence group is a Kantor-like sequence pair, as indicated by Proposition (1.11.3).

In what follows in this section, we identify $G_\pm$ with $G_\pm(\Phi)$, except if we indicate that we will use the $\Phi$-group structure. Note that this is not deceiving if we want to use the module structure because $G_\pm(K) = G_\pm(\Phi) \otimes K$ as a $\Phi$-module. We do this because we will construct a 5-graded Lie algebra $L$ of which all non 0-graded elements correspond to the modules $G_\pm(\Phi)$. We will only need the $\Phi$-group structure to prove certain statements about this Lie algebra.

**Definition 4.2.3.** Suppose that $G$ is a Kantor-like (weak) sequence pair and that $U$ is its universal representation, then we call $(G_-)_1 \oplus (G_+)_1 \leq U$ with operation

$$V_{x,y}(z) = [[x_1, y_1], z_1] = [\mathrm{ad}^{(1)}_x, \mathrm{ad}^{(1)}_y](z_1) = -\mathrm{ad}^{(1)}_z \mathrm{ad}^{(1)}_x(y_1),$$

**the Kantor pair associated with** $G$. This operation is internal since the right hand side is the $(1,1)$-linearization of $-Q^1_x(y)$ with respect to $x$.

We proceed by constructing the TKK Lie algebra. In fact, we already have all the ingredients. We consider the associated Kantor pair[1] $(G_-)_1 \oplus (G_+)_1$ in the universal representation and a Lie algebra $M$ acting on it, namely $M \cong H^1_- \oplus \mathbf{InDer}(G) \oplus H^1_+$ which we identify with the elements of the adjoint representation. We define $\mathbf{InDer}(G)$ to be the linear span of the operators $V_{x,y} = [\mathrm{ad}^{(1)}_x, \mathrm{ad}^{(1)}_y]$ with $x \in G_\pm, y \in G_\mp$ (where we also allow $x, y$ to be elements of $G/H^1_\pm$ instead of the full groups, for notational convenience). And $h \in H^1_\pm$ is identified with $\mathrm{ad}^{(2)}_h$. We take in fact a quotient of the just constructed Lie algebra, as we assume that if $V_{x,y}u_1 = V_{a,b}u_1$ for all $u \in G_\pm$ then we say that $V_{x,y} = V_{a,b}$. We did just say that it is a Lie algebra, now we prove it.

[1]In fact, we could just see it as a sign graded Lie triple system, and take the standard embedding. This would yield the same result as the construction of this section. However, the close relation of the construction, as executed in this section, will make it very easy to define the action of the universal sequence pair representation on the TKK Lie algebra. Moreover, the construction of the TKK Lie algebra in this section is applicable to all sequence pairs such that $H^1 = [G, G]$.

**Lemma 4.2.4.** ***InDer***$(G)$ *is a Lie algebra.*

*Proof.* It is sufficient to prove that $[V_{x,y}, V_{u,v}] \in \mathbf{InDer}(G)$, for $x, u \in G_+, y, v \in G_-$, since they are part of the, necessarily associative, endomorphism algebra of a $\Phi$-module. This easily follows from the following claim:

$$[[\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}], [\mathrm{ad}_u^{(1)}, \mathrm{ad}_v^{(1)}]] = [\mathrm{ad}^{(1)}_{V_{x,y}u_1}, \mathrm{ad}_v^{(1)}] + [\mathrm{ad}_u^{(1)}, \mathrm{ad}^{(1)}_{V_{x,y}v_1}] =: V_{V_{x,y}u,v} - V_{V_{x,y}v,u},$$

which we will prove now. First, we note that $\mathrm{ad}^{(1)}_{V_{x,y}u} = [\mathrm{Ad}_x^{(1)}, \mathrm{Ad}_y^{(1)}](\mathrm{ad}_u^{(1)})$ where $\mathrm{Ad}_x^{(\cdot)}$ denotes the adjoint representation of the adjoint representation $\mathrm{ad}_x^{(\cdot)}$, as $V_{x,y}(u_1) = [\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}](u_1)$ holds in the universal representation. Now, we note that

$$[\mathrm{Ad}_x^{(1)}, \mathrm{Ad}_y^{(1)}](\mathrm{ad}_z^{(1)}) = [[\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}], \mathrm{ad}_z^{(1)}],$$

for all $z$. Which proves the claim if we first use that $[\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}]$ is a derivation. ■

**Lemma 4.2.5.** *The algebra $M = H^1_- \oplus$* ***InDer***$(G) \oplus H^1_+$ *is a Lie algebra.*

*Proof.* Once again, it is sufficient to prove that the brackets are internal, since this is a submodule of an associative algebra. There are two types of brackets that need to be checked, namely those of the form $[H^1_-$,**InDer**$(G)]$ or $[H^1_-, H^1_+]$.

This can easily be proved if we identify each element with $[\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}]$ for some $x$ and $y$, which we can also do for $H^1_\pm$ since $H^1_\pm = [G_\pm, G_\pm]$. So we compute

$$[l, [\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}]] = [[l, \mathrm{ad}_x^{(1)}], \mathrm{ad}_y^{(1)}] + [\mathrm{ad}_x^{(1)}, [l, \mathrm{ad}_y^{(1)}]],$$

for $l \in M$. Note that the proof of the previous lemma shows that if $l = V_{a,b}$, then we have $[\mathrm{ad}^{(1)}_{V_{a,b}x}, \mathrm{ad}_y^{(1)}] + [\mathrm{ad}_x^{(1)}, \mathrm{ad}^{(1)}_{V_{a,b}y}]$, where we let $V_{a,b}$ act as $[\mathrm{Ad}_a^{(1)}, \mathrm{Ad}_b^{(1)}]$ on $H^1_\sigma$, which proves that brackets of the form $[H^1_-$,InDer$(G)]$ are internal. The other type of brackets can analogously be proved to be internal as well. Specifically, one shows that

$$[\mathrm{ad}_h^{(2)}, [\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}]] = [\mathrm{ad}^{(1)}_{Q_h^1(x)_1}, \mathrm{ad}_y^{(1)}] - [\mathrm{ad}^{(1)}_{Q_h^1(y)_1}, \mathrm{ad}_x^{(1)}] = V_{Q_h^1(x)_1,y} - V_{Q_h^1(y)_1,x},$$

by making use of the fact that $[\mathrm{ad}_h^{(2)}, \mathrm{ad}_x^{(1)}] = \mathrm{Ad}_h^{(2)}(\mathrm{ad}_x^{(1)}) = \mathrm{ad}^{(1)}_{Q_h^1(x)}$. ■

Now we want to prove that the Kantor part, together with $M$ forms a Lie algebra.

**Proposition 4.2.6.** *The algebra $L = H^1_- \oplus (G_-)_1 \oplus$* ***InDer***$(G)$ *$\oplus (G_+)_1 \oplus H^1_+$ is a Lie algebra.*

*Proof.* We only need to check the Jacobi identity. If three elements are in $M \leq L$ then we know that this is the case. If, only two elements are in $M$ and one is in the Kantor pair part, then we are done since we define $[f, x] = f(x)$ for $f \in M$ and $x$ in the Kantor pair part, and since $[f, g] = fg - gf$ for $f, g \in M$. When there are three elements in the Kantor pair part, it is also trivial, as the Jacobi identity holds in the universal representation and since all computations can be done there (since $[[x_1, y_1], z_1] = [\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}](z_1) =: [[\mathrm{ad}_x^{(1)}, \mathrm{ad}_y^{(1)}], z_1]$).

At last, we check whether the Jacobi identity holds when only two elements are in the Kantor pair part. This is, in essence the subject of the previous two propositions, since we saw each element of $M$ already as a bracket of two elements of the Kantor pair part, and proved that these operations are internal, by proving that $[f, [x_1, x_2]] = [f(x_1), x_2] + [x_1, f(x_2)]$. ■

We stress that $h \in H^1_\pm$ seen as an element of $L$ can be interpreted as an element of the universal representation and as an element of the adjoint representation on the universal representation, depending on the context. Specifically, if we interpret $h_2$ to be an element of the universal representation, then we get $[h_2, g_1] = \mathrm{ad}^{(2)}_h(g_1) = [\mathrm{ad}^{(2)}_h, g_1]$. Additionally, equalities making use of these different interpretations can often be carried over to the other context. When we do that, we will explain why the equalities carry over. Even the elements $V_{x,y}$ can be seen as elements of the universal representation, modulo some identifications on these elements. Specifically, we can often think of $V_{x,y}$ as $[x_1, y_1]$ (notice, however, that there can be a lot of different $[u_1, v_1]$ associated with the same $V_{x,y}$, so some caution is required).

Now, we want to determine the TKK representation in the endomorphism algebra of $L$. We can do this by making the universal representation act on this Lie algebra. So, we consider the following central cover of $L$, namely $\tilde{L} = H^1_- \oplus (G_-)_1 \oplus N \oplus (G_+)_1 \oplus H^1_+$ with $N$ the submodule generated by $[x_1, y_1]$ for $x \in G_\pm, y \in G_\mp$. Analogous to Lemma (4.2.4)[2], one shows that $N$ is a Lie subalgebra of $U$. This cover is contained in the universal representation.

There is a natural action of the universal representation on this central cover. Namely, we use

$$A = m \circ (\mathrm{Id} \otimes S) \circ \Delta,$$

with $m(a \otimes b)(c) = acb$ which maps from the universal representation to its endomorphism algebra. After the following remark, we prove that $A$ induces a representation in the endomorphism algebra of $\tilde{L}$.

*Remark* 4.2.7. Notice that this $A$ coincides with the adjoint representation. Specifically, it is the unique mapping of the universal representation into its endomorphism algebra corresponding to the adjoint representation. As such, we can use some results about the adjoint representation of sequence groups. Namely, the moreover part of Lemma (2.1.9) indicates that

$$A(x_n)(ab) = \sum_{i+j=n} A(x_i)(a)A(x_j)(b).$$

As such, we realize that

$$A(c)(ab) = \sum A(c_i')(a)A(c_i'')(b),$$

for all $c$, for some $c_i', c_i''$ determined by $\Delta(c) = \sum c_i' \otimes c_i''$. Since the universal representation is cocommutative, we learn that

$$A(c)[a, b] = \sum [A(c_i')(a), A(c_i'')(b)].$$

**Lemma 4.2.8.** *Let $U$ be the universal representation of a Kantor-like sequence pair $G$, then $A(U)(\tilde{L})$ is contained in $\tilde{L}$.*

*Proof.* We remark, as Remark (4.2.7) indicates, that it is sufficent to show that

$$A(U)((G_-)_1 \oplus (G_+)_1) \subset \tilde{L},$$

as $(G_-)_1 \oplus (G_+)_1$ generates $\tilde{L}$. Since $A$ is an algebra morphism, it is sufficient to check it for generators $x_n$, for $x \in G_\sigma$, acting on $y_1$ for $y \in G_{\pm\sigma}$. If the signs of $x$ and $y$ are opposite, then this follows exactly form restrictions (2.3), (2.4) and (2.5). If they are the same, it requires an argument. We see that $(ty)^{sx^{-1}}_{n+1} = s^n t A_{x_n}(y_1) +$ lower order terms in $s$. On the other hand, we know that $(ty)^{sx^{-1}} = (ty_1, t^2 y_2 + st[x, y])$ in the sequence $\Phi$-group. So, we see that $A_{x_1}(y_1) = [x_1, y_1] \in \tilde{L}$ and that $A_{x_n}(y_1) = 0$ for $n \geq 2$ since there are no terms belonging to $s^n t$ in $(ty_1, t^2 y_2 + st[x, y])_n$, as all terms $a$ in $(ty_1, t^2 y_2 + st[x, y])_{n+1}$ do necessarily satisfy $\deg_t(a) \geq \deg_s(a)$. ■

[2] It is less convoluted, as we do not have to lift our computations to the adjoint representation.

Now, we just need to check whether the action is trivial on the kernel of the projection of $\tilde{L}$ onto $L$. Suppose that $V_{x,y}$ has trivial action, then we get $u_1 \cdot V_{x,y} = [u_1, [x_1, y_1]] = -V_{x,y}u_1 = 0$. Additionally, we see that $\mathrm{Ad}_u^{(2)}(V_{x,y}) = 0$ in the adjoint representation of the adjoint representation. However, if $\mathrm{ad}_u^{(2)}([x_1, y_1]) = h_2$ and if we apply $A$, then we get $\mathrm{Ad}_u^{(2)}(V_{x,y}) = \mathrm{ad}_h^{(2)}$. Since the action of $H_\pm^1$ is faithful, this means that $h = 0$. So the action on the kernel is of the projection of $\tilde{L}$ onto $L$ onto $L$ is trivial.

We will denote $L$ as TKK($G$) or, in reference to what we will do later, as TKK($G, \mathbf{InDer}(G)$).

*Remark* 4.2.9. The constructed representation has some special properties. Namely, we have

$$x_n \cdot [a, b] = \sum_{i+j=n} [x_i \cdot a, x_j \cdot b]$$

for all $a, b \in L$ and $n \in \mathbb{N}$. We argued why this is the case in Remark (4.2.7). Note, moreover, that this is a sequence pair representation, as we just need to let $U \otimes K$ (with $U$ the universal representation) act on $\tilde{L} \otimes K$ using the adjoint representation, and then divide out by $Z(\tilde{L} \otimes K) = Z(\tilde{L}) \otimes K$ which gets mapped to itself under the action of $U \otimes K$.

There are 2 equations that need to hold in order to add a grading element. We first prove that these are satisfied for sequence $\Phi$-groups. We remark that if we were not working with sequence $\Phi$-groups, one would still be able to prove that the possibility to add a grading element is equivalent to these equations.

**Lemma 4.2.10.** *Let $G$ be a sequence $\Phi$-group, and $\rho$ a sequence $\Phi$-group representation. Each $x \in G$ satisfies*

- $3(x_3 - x_1x_2) + x_1^3 = 0,$
- $4x_4 - 2x_2^2 + x_1x_3 - 2x_3x_1 + x_2x_1^2 = 0.$

*Proof.* We work in $G(K)$ with $K = \Phi[t]/(t^2 - t - 1)$. The first equation follows from computing the third coordinate of $(t \cdot x)((1-t) \cdot x)(-1 \cdot x) \in H^1$. We compute this explicitly. We see that

$$(t \cdot x)((1-t) \cdot x) = (1, x_1, 3x_2 - x_1^2, 4x_3 - x_1x_2, \ldots),$$

so when we multiply this with $-1 \cdot x$, we get

$$(1, 0, 4x_2 - 2x_1^2, 3x_3 - 3x_1x_2 + x_1^3, \ldots),$$

where we needed in that $[x_1, x_2] = 0$ in order to get both expressions. Note that all representations are assumed to be even, so we get the first equality.

The second equality can be proved, by comparing the fourth coefficients of $t \cdot_H (x(-x)) \times (1 - t) \cdot_H (x(-x))$ and $x(-x)$ (where the scalar multiplications $\cdot_H$ with $t$, $1-t$ are done in $H^1$), as they should equal. ∎

**Theorem 4.2.11.** *Let $G$ be a (weak) sequence pair, and $L$ a 5-graded Lie algebra such that there is a defining (weak) sequence pair representation of $G$ in the endomorphism algebra of $L$. Suppose, moreover, that $G_\sigma \cong L_\sigma \oplus L_{2\sigma}$ with some multiplication $(a, b)(c, d) = (a + c, b + d + \psi_\sigma(a, c))$ for bilinear forms $\psi_\sigma$, in such a way that $x_1(y) = [x, y]$ for $x \in L_\sigma$ and $x_2(y) = [x, y], x_1(y) = 0$ for $x \in L_{2\sigma}$, and for all $y \in L$. Then, we can add a grading element.*

*Proof.* First, we extend the action of the sequence groups $G_\pm$ on $\zeta$. We note that the possible actions on $\zeta$ are restricted. Namely, we can identify $\zeta$ with the element $2e_2 + e_1 - e_{-1} - 2e_{-2}$ in the endomorphism algebra of $L$, where $e_i$ are the projection operators $e_i : L \longrightarrow L_i$, as such the action $\mathrm{ad}_x^{(n)}(2e_2 + e_1 - e_{-1} - 2e_{-2})$ should correspond to an inner derivation. However, this does not necessarily determine the element in $L$ uniquely since there can be multiple elements with the same inner derivation. We determine the unique sensible element.

We can prove that for any element $e$ of the endomorphism algebra the following equation holds[3]

$$\mathrm{ad}_x^{(n)}(e) \cdot u = x_n \cdot e(u) - \sum_{\substack{i+j=n \\ i \neq n}} \mathrm{ad}_x^{(i)}(e) \cdot (x_j \cdot u),$$

for all $n$. In particular, it will hold for the grading element $\zeta$. We only prove it for $\zeta$. It holds by definition for $n = 1$. Next, we prove it for $n = 2$, by calculating

$$\begin{aligned} \mathrm{ad}_x^{(2)}(\zeta) \cdot u &= (x_2\zeta - x_1\zeta x_1 + \zeta(x^{-1})_2) \cdot u \\ &= x_2 \cdot [\zeta, u] - x_1 \cdot [\zeta, x_1 \cdot u] + [\zeta, (x^{-1})_2 \cdot u] \\ &= x_2 \cdot [\zeta, u] - [x_1 \cdot \zeta, x_1 \cdot u] - [\zeta, x_2 \cdot u], \end{aligned}$$

where the last equality holds because $x_2 + (x^{-1})_2 = x_1^2$ for all $x$ and $[x_1, \zeta] = x_1\zeta - \zeta x_1$, since $0 = (xx^{-1})_2 = x_2 + (x^{-1})_2 - x_1^2$

For $n = 3, 4$ the calculation is similar using strong induction, but we will not only need $x_2 + (x_2)^{-1} = x_1^2$. We will also need $x_3 + x_3^{-1} = x_2x_1 - x_1x_2^{-1}$ and $x_4 + x_4^{-1} = x_3x_1 - x_1x_3^{-1} - x_2x_2^{-1}$. We also used suggestive notation $[\zeta, y]$ for the action of $\zeta$. This does, however, not indicate any assumptions on the interaction of $\zeta$ with elements $x_n$ even if we write $[x_n \cdot \zeta, y]$. For $n = 5$ this does not matter any more, since $x_5 \cdot \zeta = 0$ by the grading.

We start by determining what is necessary (and sufficient) for $x \in G_+$ to ensure that $x_n \cdot \zeta$ is an inner derivation. For $x_1$ this is trivial, as $x_1 \cdot \zeta$ is, and should be, $-x_1 \in L$. For $x_2$ we set $x_2 \cdot \zeta = (x_1^2 - 2x_2)$. Firstly, this form is necessary by

$$x_2 \cdot [\zeta, u] = [x_2 \cdot \zeta, u] - [x_1, [x_1, u]] + [\zeta, x_2 \cdot u],$$

which leads to

$$-2x_2 \cdot u + x_1^2 \cdot u = [x_2 \cdot \zeta, u].$$

We still need to prove that $x_2 \cdot \zeta \in L$. We compute that $x(-x) = (1, 0, 2x_2 - x_1^2, 0, \ldots)$. Since there is a scalar multiplication on $H_+^1$, we see that $(x_1^2 - 2x_2) \in L$.

Now, we check the higher coordinates. We prove that the equations which are necessary for

$$x_3 \cdot \zeta = 0, \quad x_4 \cdot \zeta = 0$$

are exactly the equations of Lemma (4.2.10). We get

$$x_3 \cdot [\zeta, y] = [x_3 \cdot \zeta, y] + [x_2 \cdot \zeta, x_1 \cdot y] - [x_1, x_2 \cdot y] + [\zeta, x_3 \cdot y].$$

If we assume that $y$ is either $-1$ or $-2$ graded, we get

$$-3x_3 \cdot y = (x_1^2 - 2x_2)x_1 \cdot y - x_1x_2 \cdot y + [x_3 \cdot \zeta, y].$$

[3] There are no assumptions related to the invertibility of scalar. Specifically, what this equation says, is that $\exp(x)$ is an automorphism if $\mathrm{ad}_x^{(5)}(\mathrm{ad}\ y) = 0$ and $\mathrm{ad}_x^{(6)}(\mathrm{ad}\ y) = 0$ for all $y \in L$. We already hinted at this in Remark (2.4.9).

Thus, we get

$$-[x_3 \cdot \zeta, y] = 3x_3 \cdot y - 2x_2x_1 \cdot y - x_1x_2 \cdot y + x_1^3 \cdot y.$$

This means that $3(x_3 - x_1x_2) + x_1^3$ must act as 0 on $L$, since we can assume that $x_1$ and $x_2$ commute as $[x_1, x_2] = x(-x)_3 = 0$. So, $x_3$ interacts nicely with $\zeta$ if and only if the first equation is satisfied. Similarly, one can check that $x_4 \cdot \zeta = 4x_4 - x_3x_1 - 2x_2^2 + x_2x_1^2$.

This definitely extends the action of both groups to $L$ together with the grading element. Since $\zeta$ was identified with an element in the endomorphism algebra of $L$, the restrictions which make this pair of sequence groups a sequence pair remain unchanged. ■

**Corollary 4.2.12.** *Each (weak) Kantor-like sequence pair $G$ has a weak sequence pair representation as Lie exponentials in the endomorphism algebra of a Lie algebra*

$$\mathit{TKK(G)} = H_-^1 \oplus (G/H^1)_- \oplus L_0 \oplus (G/H^1)_+ \oplus H_+^1,$$

*with grading element contained in $L_0$. Moreover if $1/6 \in \Phi$, each (weak) Kantor-like sequence pair has a defining sequence pair representation.*

*Proof.* We only need to prove the moreover part. We will prove that the earlier constructed TKK representation is a sequence pair representation. We want to prove that the $(1, [a, b], [2a, 2b], \ldots)$ are elements of the groups for coprime $a, b$ different from 1. We note that we can apply Theorem (2.3.3), by Remark (4.2.9) on the relation

$$a\Delta \sum b_i \otimes c_i \quad \text{if and only if } a[u, v] = \sum [b_iu, c_iv].$$

The $[a, b]$ are exponentials and are thus, by Lemma (2.4.13), contained in the groups $G_+, G_-$. ■

**Corollary 4.2.13.** *For each Kantor pair $P = (P_\pm, V^\pm)$ over $\Phi$ with $1/30 \in \Phi$, there is a unique (weak) Kantor-like sequence pair $G$ such that*

$$(G/H^1)_\sigma = P_\sigma,$$

*and that*

$$[ad_x^{(1)}, ad_y^{(1)}](z_1) = V^\sigma_{x_1,y_1}(z_1),$$

*for $x, z \in G_\sigma$, and $y \in G_{-\sigma}$. Moreover, we can endow the associated pair with a sequence pair structure, i.e. there exists a $Q$ satisfying (2.4) for a defining representation satisfying (2.6).*

*Proof.* The existence of such a sequence pair follows from Theorem (2.4.8). The uniqueness follows from the fact that each weak Kantor-like sequence pair, has a representation in the endomorphism algebra of a Lie algebra TKK($G$) as exponentials. We note that all these Lie algebras coincide with the TKK Lie algebra TKK($P$, **InDer** $+ \Phi\zeta$) of $P$. So, all possible (weak) Kantor-like sequence pairs have the same defining representation, which means that they are isomorphic. ■

If $1/5 \notin \Phi$ we do not know of the existence of a sequence pair corresponding to a Kantor pair. However, we do know that if it exists, then it must be unique.

## 4.3 Jordan-Kantor-like sequence pairs

**Definition 4.3.1.** A sequence pair $G$ over $\Phi$ with $1/2 \in \Phi$ is **Jordan-Kantor-like** if there exists operators

$$P_\sigma(K) : G_\sigma(K) \longrightarrow \mathrm{Hom}_{Set}(H^1_{-\sigma}(K), G_\sigma(K))$$

and a defining representation of $G$, such that for all $x \in G_\sigma(K), h \in H^1_{-\sigma}(K)$

$$P_x(h)_n = [3n, 2n], \tag{4.4}$$

and

$$[a, b] = 0, \tag{4.5}$$

hold, for $2 \neq a/b > 3/2$. A **weak Jordan-Kantor-like sequence pair** is a weak sequence pair with and additional operator $P^1_x(h) \in G/H^1_\sigma(K)$ such that $P^1_x(h)_1 = [3, 2]$. The **(weak) Jordan-Kantor-like sequence pair representations** are exactly the sequence pair representations of a Jordan-Kantor-like sequence pair that satisfy these additional restrictions.

*Remark* 4.3.2.
- The class of (weak) Jordan-Kantor-like sequence pair representations of a Jordan Kantor-like sequence pair is a sensible collection of representations. This is easily proved, using Lemmas (4.1.1), (4.1.3) and (4.1.5). It is obvious which additional relations must by divided out to construct the universal (Jordan-Kantor-like sequence pair) representation. Moreover, these relations are compatible with the $\mathbb{Z}$-grading. So, Theorem (4.1.12) proves that the universal representation is a cocommutative $\mathbb{Z}$-graded Hopf algebra.
- The TKK representation of any Jordan-Kantor pair ($1/30 \in \Phi$) is a Jordan-Kantor-like sequence pair representation.
- We chose to immediately require that the $[a, b]$ are zero instead of saying that $\mathrm{ad}^{(i)}_x(y_j) = \sum_{t \in U} t$ for some $U$. Note that these are equivalent. This can be proved analogously to Lemma (2.4.6).
- If $\Phi$ is a field of characteristic different from 2 and 3, we will see that each sequence pair representation of a Jordan-Kantor-like sequence pair is a Jordan-Kantor-like sequence pair representation.
- Later, in Definition (8.2.1), we will reconsider what the Jordan-Kantor-like sequence pairs exactly are. This will allow us to introduce them if $1/2 \notin \Phi$. The new notion may be a bit more restrictive if $1/3 \notin \Phi$. Restrictions (4.4) and (4.5) are still part of the definition. We will make an additional assumption so that we can guarantee that there is a sensible notion of an inner derivation algebra and a sensible action on the TKK Lie algebras.

In this section, we also identify $G_\pm$ with $G_\pm(\Phi)$. Consider the Lie subalgebra $\tilde{L}$ of the universal representation $U$, given by

$$\tilde{L} = (H^1_-)_2 \oplus (G_-)_1 \oplus L_0 \oplus (G_+)_1 \oplus (H^1_+)_2,$$

where $L_0$ is the submodule spanned by the $[a_1, b_1], [g_2, h_2]$ for $a \in G_+, b \in G_-, g \in H^1_+, h \in H^1_-$. Brackets of the kind $[(H^1_\sigma)_2, (G_{-\sigma})_1] \in \tilde{L}$ are internal. Specifically, we know that $[g_2, b_1] = Q^1_g(b)_1$. All other brackets are trivially internal, or are internal if $[L_0, \tilde{L}] \subset \tilde{L}$. This is easily checked using the operator $Q^1$, the $(1, 1)$-linearization of $Q^1$ and the $(2, 2)$-linearization of $Q^2$.

**Lemma 4.3.3.** *There is a (weak) Jordan-Kantor-like sequence pair representation in the endomorphism algebra of $\tilde{L}$, defined as $x \mapsto (Id, \ldots, A(x_n)_{|\tilde{L}}, \ldots)$. Moreover, this representation can be extended so that $\tilde{L}$ contains a grading element.*

*Proof.* Analogous to Kantor-like sequence pairs, the action is internal if we let $x \in G_+$ act upon $(G_-)_1 \oplus (G_+)_1 \oplus (H^1_+)_2$. Only for the action upon an $h$ in $L_0$ or $(H^1_-)_2$, we need to see why the action is internal. We assume that the representation is in standard form and that $x = (x_1, s)$, so that

$$x_2 \cdot h = [x_1, [x_1, h]]/2 + [s, h_2] = -[x_1, Q^1_h(x)_1]/2 + [s, h_2],$$

$x_3 \cdot h = P^1_x(h)_1$ and $x_4 \cdot h = Q^2_x(h)_2$. As a consequence, $x$ also maps $L_0$ to $\tilde{L}$ since $L_0$ is generated by all the other parts of $\tilde{L}$, upon which $x$ acts well.

By Theorem (4.2.11), we can add a grading element compatible with the sequence pair representation. ∎

Consider $L'_0$, the quotient of $L_0$ (with grading element) by setting $l = m$ if they act the same on

$$(H^1_-)_2 \oplus (G_-)_1 \oplus (G_+)_1 \oplus (H^1_+)_2.$$

Then

$$L = (H^1_-)_2 \oplus (G_-)_1 \oplus L'_0 \oplus (G_+)_1 \oplus (H^1_+)_2$$

is a Lie algebra. We note that this is $\tilde{L}/Z(\tilde{L})$ since 2 is invertible and since there is a grading element. Suppose that $x_n \cdot z$ with $z \in Z(L)$ is not contained in $Z(L)$. Then $n$ is necessarily equal to 2. Note that $x_2 = x_1^2/2 + s$, i.e. it is a polynomial in inner derivations, which map $Z(L)$ to 0.

We call the representation of a Jordan-Kantor-like sequence pair in the endomorphism algebra of $L$ the **TKK-representation.** We note that if $1/6 \in \Phi$ the construction of the TKK Lie algebra coincides with the usual TKK representation of the associated[4] Jordan-Kantor pair

**Corollary 4.3.4.** *For each Jordan-Kantor pair $P$ over $\Phi$ with $1/30 \in \Phi$, there exists a unique (weak) Jordan-Kantor-like sequence pair $G$ with associated Jordan-Kantor pair $P$. Moreover, if $1/6 \in \Phi$, then each weak Jordan-Kantor-like sequence pair has a defining Jordan-Kantor-like sequence pair representation.*

*Proof.* We only need to prove the moreover part. The action on the TKK Lie algebra induced by the Hopf algebra satisfies $x_n \cdot [a, b] = \sum_{i+j=n} [x_i \cdot a, x_j \cdot b]$ for all $n$. So, we can apply the corresponding either part of Theorem (2.4.8). ∎

**Corollary 4.3.5.** *If $1/6 \in \Phi$, then each sequence pair has a defining Jordan-Kantor-like sequence pair representation.*

*Proof.* We prove that each sequence pair representation is a weak Jordan-Kantor-like sequence pair representation. In that case, the corollary follows from Corollary (4.3.4). We compute, for $x \in G_\pm(K), h \in H^1_\mp(K)$, that

$$[3, 2] = \mathrm{ad}^{(3)}_x(h_2) = \mathrm{ad}^{(2)}_x \mathrm{ad}^{(1)}_x(h_2) - 1/3(\mathrm{ad}^{(1)}_x)^3(h_2) = -Q_x Q_h(x)_1 + 1/3 Q^{1,(1,1)}_{x,x} Q_h(x)_1$$

with $Q^{1,(1,1)}$ the $(1,1)$-linearization of $Q^1$. This is definitely an element of $G/H^1_\pm(K)$. So, we get that the operator $P^1$ exists. ∎

[4] We do not write out how you can get all the operators of the Jordan-Kantor pair, but you can clearly derive them.

## 4.4 Hermitian structurable algebras

Consider an associative algebra $A$ with involution $a \mapsto \bar{a}$, a right $A$-module $M$ and a hermitian form $h : M \times M \longrightarrow A$. We define $B = A \oplus M$, the **hermitian structurable algebra**, with multiplication

$$(a, m_1)(b, m_2) = (ab + h(m_2, m_1), m_2 a + m_1 \bar{b}),$$

and involution

$$\overline{(a, v)} = (\bar{a}, v).$$

As was the case for associative structurable algebras, we will, at least if $1/2 \notin \Phi$, not be able to give an exhaustive description. If we speak about such algebras, we suppose that there is a quadratic form $f$, as it was the case for associative algebras, which polarizes to $x\bar{y} - \psi(x, y)$ for a bilinear form $\psi : B \times B \longrightarrow S$ with $S$ the image of $x \mapsto x - \bar{x}$.

*Example* 4.4.1. (Hermitian structurable algebras $1/2 \in \Phi$) We construct sequence pairs related to a specific subclass of hermitian structurable algebras over $\mathbb{Z}[1/2]$ such that every Hermitian structurable algebra is a quotient of one. We consider a free unital associative algebra $A$ with three sets of generators, namely

$$H_{gen}, S_{gen}, \{h(i, j) | i, j \in I\}$$

for some indexing set $I$. There is a unique involution on $A$ defined as the identity on $H$, $-\mathrm{Id}$ on $S$ and maps $h(i, j) \mapsto h(j, i)$. Note that each associative algebra with involution, if $1/2 \in \Phi$, is a quotient of such an algebra (even without the generators $h(i, j)$). We also consider a free right $A$-module $M$ with basis $(m_i)_{i \in I}$. There is a unique hermitian form $h$ such that $h(m_i, m_j) = h(i, j)$, which is the hermitian form we will consider.

Now, we consider $B = A \oplus M$ with earlier specified multiplication, this falls under the hermitian structurable algebras. Note that every hermitian structurable algebra over a ring $\Phi$ with $1/2$ is a quotient of such an algebra. So, the question is whether we can also define the operators $Q^1, T, Q^2$ in this more general context. We will make use of the structure of what should be the TKK Lie algebra. Specifically, we use that structure to determine the operators, and then we will argue (using $B \otimes \mathbb{Q}$) why this gives us the structure of a sequence pair. Since we have assumed that $1/2 \in \Phi$ we do not need to bother with $Q^1$, as we know that it should be $Q^1_{x'}(y') = V_{x',y'}x/2 + zy \in G/H^1(K)$ with $x' = (x, z) \in (B \times S) \otimes K$ and $y' = (y, w)$. We also know that $T$ is uniquely determined if $1/3 \in \Phi$, using $x'_3 = x'_1 x'_2 - {x'_1}^3/3$ and that $T_{x'}(y') = x'_3 \cdot y$ in the TKK representation. We will prove that ${x'_1}^3 \cdot y = x^3 \cdot y$ is divisible by 3 over $\mathbb{Z}[1/2]$, which is a domain, so that the corresponding formula is unique.

One can compute $x^3 \cdot y = [V_{x',y'}x, x]$ in the TKK representation, or at least what it should be. So, set $x = (a, u)$ and $y = (b, v)$. We get

$$V_{x',y'}x = 2(x\bar{y})x - (x\bar{x})y,$$

and

$$[V_{x',y'}x, x] = 2((x\bar{y})x)\bar{x} - ((x\bar{x})y)\bar{x} - 2x(\bar{x}(y\bar{x})) + x(\bar{y}(x\bar{x})).$$

We compute these terms one by one. First, we compute

$$\begin{aligned}((x\bar{y})x)\bar{x} =&((a\bar{b}+h(v,u), va+ub)x)\bar{x}\\ =&(a\bar{b}a+h(v,u)a+h(u,v)a+h(u,u)b, ua\bar{b}+uh(v,u)+va\bar{a}+ub\bar{a})\bar{x}\\ =&(a\bar{b}a\bar{a}+h(v,u)a\bar{a}+h(u,v)a\bar{a}+h(u,u)b\bar{a}\\ &+h(u,u)a\bar{b}+h(u,u)h(v,u)+h(u,v)a\bar{a}+h(u,u)b\bar{a}\\ &, ua\bar{b}a+uh(v,u)a+uh(u,v)a+uh(u,u)b\\ &+ua\bar{b}a+uh(v,u)a+va\bar{a}a+ub\bar{a}a)\\ =&(a\bar{b}a\bar{a}+h(v,u)a\bar{a}+2h(u,v)a\bar{a}+2h(u,u)b\bar{a}+h(u,u)h(v,u)+h(u,u)a\bar{b},\ldots),\end{aligned}$$

where we dropped the last coordinate, as this will, necessarily, be zero in

$$2((x\bar{y})x)\bar{x}+x(\bar{y}(x\bar{x}))-\overline{2((x\bar{y})x)\bar{x}+x(\bar{y}(x\bar{x}))},$$

which is what we are computing.

We calculate the value of

$$\begin{aligned}x(\bar{y}(x\bar{x})) =&x(\bar{y}(a\bar{a}+h(u,u), 2ua))\\ =&x(\bar{b}a\bar{a}+\bar{b}h(u,u)+2\bar{a}h(u,v), 2ua\bar{b}+va\bar{a}+vh(u,u))\\ =&(a\bar{b}a\bar{a}+a\bar{b}h(u,u)+2a\bar{a}h(u,v)+2b\bar{a}h(u,u)+a\bar{a}h(v,u)+h(u,u)h(v,u)\\ &, 2ua\bar{b}a+va\bar{a}a+vh(u,u)a+ua\bar{a}b+uh(u,u)b+2uh(v,u)a)\\ =&(a\bar{b}a\bar{a}+a\bar{b}h(u,u)+2a\bar{a}h(u,v)+2b\bar{a}h(u,u)+a\bar{a}h(v,u)+h(u,u)h(v,u),\ldots).\end{aligned}$$

We combine those to compute $2((x\bar{y})x)\bar{x}+x(\bar{y}(x\bar{x}))$. This yields

$$\begin{aligned}2((x\bar{y})x)\bar{x}+x(\bar{y}(x\bar{x})) =&(3a\bar{b}a\bar{a}+3h(u,u)h(v,u)\\ &+4h(u,v)a\bar{a}+a\bar{a}h(v,u)+4h(u,u)b\bar{a}+b\bar{a}h(u,u)\\ &+2\cdot(h(u,u)a\bar{b}+b\bar{a}h(u,u)+a\bar{a}h(u,v)+h(v,u)a\bar{a}),\ldots),\end{aligned}$$

where we grouped the terms so that all terms $t$ are either already 3 times a term $t'$, part of $4t'+\bar{t'}$ or part of $2(t'+\bar{t'})$. As such, we obtain that

$$\begin{aligned}x^3\cdot y =&3(a\bar{b}a\bar{a}-a\bar{a}a\bar{b}+h(u,u)h(v,u)-h(u,v)h(u,u)\\ &+h(u,v)a\bar{a}-a\bar{a}h(u,v)+h(u,u)b\bar{a}-a\bar{b}h(u,u),0).\end{aligned}$$

Therefore, we know that $T$ is well defined over $\mathbb{Z}[1/2]$, as $T_{x'}(y')=[x,Q^1_{x'}(y')]-x^3\cdot y/3$. The map $Q^2$ can easily be defined by setting

$$Q^2_{x'}[u,v]=-V_{x',v}\cdot T_{x'}(u)+V_{x',u}\cdot T_{x'}(v)+[Q^1_{x'}(u),Q^1_{x'}(v)].$$

If we see $\mathbb{Q}$ as a $\mathbb{Z}[1/2]$ algebra, then for $B\otimes\mathbb{Q}$ we know that the result of this construction is a (weak) Kantor-like sequence pair. Thus, over $\mathbb{Q}$ it is a Kantor-like sequence pair. But we know that all operators involved map the result of this construction, seen as a substructure, to itself, from which we conclude that the construction applied to $B$ also induces a (weak) Kantor-like sequence pair. We can avoid the weakness assumption by making use of $1/2$. Specifically, we know that for $B\otimes\mathbb{Q}$, there exists an operator $Q$. It is sufficient to see that $Q$ maps the $\mathbb{Z}[1/2]$-substructure to itself. This is implied, where $y=(y_1,s)$ and $x=(x_1,t)$, by

$$[4,2]=\mathrm{ad}_x^{(4)}(y_2)-[1,1][3,1]=Q^2_x(s)+Q^1_x(y)^2_1/2-1/2[[1,1],[3,1]]\in Q_x(y)^2_1/2+H^1(\mathbb{Z}[1/2]).$$

*Example* 4.4.2. Without the assumption $1/2 \in \Phi$ it is not that easy. A subclass for which it is possible, are the hermitian structurable algebras $A \oplus M$ such that $A$ has a quadratic form which permits us to construct a special sequence pair from $A \oplus S$. Additionally, we need a quadratic form $M \longrightarrow A$ creating a hermitian special sequence pair. Then we consider the quadratic form $f(a, m) = (f(a) + f(m), ma)$ on the hermitian structurable algebra. This allows us to define

$$Q^1_{((a,m),s)}(y) = (f(a, m) + s)y + ((a, m)y)(a, m) \in G/H^1(\Phi).$$

As in the previous example, we can construct $T, Q^2$ from $Q^1$. This gives the structure of a weak sequence pair. It is a bit harder to see whether it is a sequence pair.

# 5 The exponential property

In this chapter, we generalize Faulkners [Fau00] approach to prove that the Hopf algebra $H$ associated to a Jordan pair $(V_+, V_-)$, with $V^+, V^-$ free $\Phi$-modules, has as primitive elements

$$V_- \oplus \mathcal{P}(\mathcal{H}) \oplus V_+.$$

Specifically, we generalize this to Jordan-Kantor-like sequence pairs over fields of characteristic different from 2 and 3. Even though the most important results of this chapter are for such fields, we will also prove some important theorems for general commutative unital rings, or such rings with $1/6$. Moreover, all results hold, in fact, if the sequence groups are formed out of free $\Phi$-modules with $1/6 \in \Phi$. Before section 5.3 we will work over commutative unital rings, while we will work over fields in that section.

## 5.1 A unique associative factorization

First, we recall some elements of the first section of [AF99].

**Definition 5.1.1.** Suppose $A$ is an associative algebra over $\Phi$ with idempotents $e$ and $f$. We call $x \in eAf$ $(e,f)$-**invertible** if there exists an $y \in fAe$ such that $xy = e$ and $yx = f$, i.e. if $x$ is invertible in the Jordan pair $(eAf, fAe)$ (with multiplication $Q_x(y) = xyx$) with inverse $y$.

We can generalize this to more idempotents. If $e = \sum_{i=1}^n e_i$ for pairwise orthogonal idempotents $e_i$ and $f = \sum_{j=1}^m f_j$ for pairwise orthogonal idempotents $f_j$, then we can write $x \in eAf$ as $\sum e_i x f_j$. For $n = m$ and $x \in eAf$ with $e_i x f_j = 0$ for $i \neq j$, we call $x$ $(E,F)$-**diagonal** (with $E = (e_1, \ldots, e_n), F = (f_1, \ldots, f_m)$). We denote the set of $(E,F)$-diagonal elements as $D_{E,F}$. Moreover, we use $U_E$ to denote the set of elements which are sums $e + \sum_{i<j} e_i x e_j$. This can be interpreted as the set of upper triangular matrices with 1 on the diagonal.

**Lemma 5.1.2.** *Using the notation of the previous definition: $U_{E^{op}} \times D_{E,F} \times U_F \longrightarrow U_{E^{op}} D_{E,F} U_F$ is a bijection.*

*Proof.* This is [AF99, Lemma 1]. ∎

**Proposition 5.1.3.** *Suppose that $H$ is a cocommutative 2-primitive $\mathbb{Z}$-graded Hopf algebra over $\Phi$ generated by the elements of positive and negative homogeneous divided power series and 1. If $x$, respectively $y$, is a positive, resp. negative, homogeneous dps in $H$, then there exist dps'es $h, u, v \in H[[s,t]]$ such that $h_i$ is 0-graded for each $i$, $u$ is a positive homogeneous dps and $v$ is a negative homogeneous dps, so that*

$$\exp(h) = \exp(v)\exp(sx)\exp(ty)\exp(u).$$

*Suppose that $L$ is the Lie algebra of primitive elements of $H$. If the action $x \mapsto A_{x|_L}$ of $H$ on $L$ is faithful, or if $1/2 \in \Phi$, then $u, v, h$ are unique.*

*Proof.* Before we start, we note that $A_x(L) \subset L$. Specifically, we know for positive and negative homogeneous divided power series that

$$\Delta(A_{x_n}(c)) = \sum_k \sum_{i+j=n} A_{x_i}(c'_k) \otimes A_{x_j}(c''_k),$$

with $\Delta(c) = \sum_k c'_k \otimes c''_k$. This is basically a more general formulation of Lemma (2.4.18). In particular, for primitive elements $c$, we obtain

$$\Delta(A_{x_n}(c)) = A_{x_n}(c) \otimes 1 + 1 \otimes A_{x_n}(c),$$

using $A_{x_n}(1) = \eta(\epsilon(x_n)) = \delta_{n0}$. We also know that $A_1 = \mathrm{Id}$. Hence $A_x(L) \subset L$ for all $x \in H$.

We first prove the unicity if there exists a faithful representation. Note that if $1/2 \in \Phi$, then we can add a grading element to $L$ and the representation would be faithful. The fact that we can add such a grading element is a consequence of Lemma (4.2.11). Note that there are natural orthogonal idempotents in the endomorphism algebra of $L$, namely the projection operators $e_i : L \to L_i$ onto the graded components, this system of idempotents will be denoted with $E$. We want to apply Lemma (5.1.2), to prove the uniqueness of $v$, $u$ and $h$. Note that $h \in D_{E,E}$ since it is 0-graded. Moreover, $u$ is a positive homogeneous dps, and as such, it is an element of $U_E$, while $v$ is an element of $U_{E^{op}}$. Therefore, it is sufficient to prove that $\exp(sx)\exp(ty)$ is an element of $U_{E^{op}} D_{E,E} U_E$.

We accomplish this by proving the first part of the theorem. Namely, we prove that

$$\exp(h) = \exp(v)\exp(sx)\exp(ty)\exp(u),$$

has a solution with $h$ 0-graded, $v$ a negative homogeneous dps and $u$ a positive homogeneous dps. Corollary (2.4.19) almost literally proves this, if we use the elements of the form $(a, b)$, $[a, b]$ defined from these divided power series, we see that

$$\exp(ty)^{\exp(sx)^{-1}} = \sum_{i,j} s^i t^j (i, j) = \prod \exp(s^i t^j [i, j]),$$

where we stress the fact that the order of the product is increasing on fractions $i/j$. Consequently, the corollary says that

$$\exp(sx)\exp(ty)\exp(sx^{-1}) = \exp(v^{-1})\exp(st[1, 1])\exp(u^{-1})\exp(sx^{-1})$$

for some $v, u$ which are a negative and positive homogeneous dps. ■

Note that we have determined the explicit form of $u, v$ and $h$ in terms of the elements $[a, b]$. So, this $u$ and $v$ are defined for all sequence pair representations. So, the question remains whether, for a sequence pair representation, the elements $u$ and $v$ are in the image of the sequence groups.

Now, we can generalize the exponential property introduced by Faulkner [Fau00, Section 6].

**Definition 5.1.4.** Let $G$ be a sequence pair over $\Phi$. Let $x \in G_+(\Phi)$ and $y \in G_-(\Phi)$. Now, we define elements $u, v \in G(\Phi[[s, t]])$ for a sequence pair representation $\rho : G \longrightarrow A$. Let $V_\sigma$ be the set of coprime $(a, b)$ for which there exists an $x(a, b) \in G_\sigma(\Phi)$ (we wrote $x(a, b)$ to stress the dependence on $a$ and $b$) such that $x_n(a, b) = [na, nb]$. We define $V'_\sigma$ as the similar set with elements such that $x'_{2n}(a, b) = [na, nb]$. Set $u = (sx) \prod_{(a,b)\in V_+} s^a t^b x(a, b) \prod_{(a,b)\in V'_+} s^{2a} t^{2b} x'(a, b)$ in increasing order on $a/b$. We define $v$ as the similar element contained in $G_-(\Phi[[s, t]])$.

If

$$\sum h_{pq}s^p t^q = h = \exp(v)^{-1}\exp(sx)\exp(ty)\exp(u)^{-1},$$

satisfies $h_{pq} = 0$ if $p \neq q$, then we say that $\rho$ has **the exponential property**, which we will shortly denote as $E$. If it only satisfies $h_{pq} = 0$ for $p \neq q$ when $\min(p, q) \leq N$ then we say that it satisfies the **restricted exponential property** [1] $E_N$.

*Remark* 5.1.5. Note that the exponential property is equivalent with requiring that all $\exp[a, b]$ with $a \neq b$ are contained in a group $G_\pm(\Phi)$. The restricted exponential property is a bit harder to gauge. Intuitively it **should** say that all $\exp[a, b]$ with $a$ or $b$ low enough should be contained in the groups. However, this does not correspond to the technical definition, since the first $k$ terms of $\exp[a, b]$ could be zero while the $k + 1$-th term is non-zero and as such even for low $a, b$ we do not know whether $\exp[a, b]$ is contained in $G_\pm(\Phi)$ if the restricted exponential property holds.

To avoid the related problems, we will slightly adjust the exponential property later. This is achieved through fixing sensible $u$ and $v$ for (Jordan-Kantor-like) sequence pairs and all representations. The restricted exponential property will behave better.

**Corollary 5.1.6.** *Suppose $H$ is a cocommutative $2$-primitive $\mathbb{Z}$-graded Hopf algebra over $\Phi$ generated by the elements of positive and negative homogeneous dps'es. The sequence pair associated to $H$ satisfies the exponential property.*

*Proof.* This is true by definition. ∎

## 5.2 The TKK representation satisfies the exponential property

We assume that $1/6 \in \Phi$. Before we can prove that the TKK representation satisfies the exponential property, we need to carry out some preliminary investigations. We assume that the sequence group representations are in standard form.

**Definition 5.2.1.** Note that $B = \sum (st)^i[i, i]$ for $x \in G_\pm(\Phi)$ and $y \in G_\mp(\Phi)$ is well defined, even if the TKK representation does not satisfy the exponential property. We call the action of $B$ on the groups $G_+$ and $G_-$, by making $B$, as an element of the endomorphism algebra of the TKK Lie algebra, act on the Lie algebra, the **Generalized Bergman operator**. The operator associated with this action will be denoted as $B_{sx,ty}$, we will drop $sx, ty$ whenever they are obvious from the context. We call the $u$ and $v$ of the definition of the exponential property, the **right and left quasi-inverse**, if they exist in the sequence groups and satisfy

$$B = \exp(v)^{-1}\exp(sx)\exp(ty)\exp(u)^{-1}.$$

*Remark* 5.2.2. It is possible to define the Generalized Bergman operator broader than here, where we defined it using formal power series. Nonetheless, the used definition achieves what we need to achieve. The names 'generalized Bergman operator' and 'right and left quasi-inverse' are chosen to reflect the fact that we used the equivalency of Theorem (1.7.9) to define these notions. It is remarkable that we know for Jordan pairs that $[n, n] = 0$ for $n \geq 3$. Specifically, $B(sx, ty) = 1 + st[1, 1] + s^2t^2[2, 2] + \ldots$ should act as the usual bergman operator $1 + stD_{x,y} + s^2t^2Q_xQ_y$.

[1] This is not an easy definition to work with, but it has the now needed flexibility. Specifically, the fact that $u$ and $v$ depend on the representation is something which, if possible, should be avoided. After the investigations of the next section, we will be able to formulate (if $1/2 \in \Phi$) a slightly different version of the exponential property for which the restricted exponential property is better suited for inductive arguments.

In the following lemma, we do not really display all the computations, but merely give a recipe to determine what the ideal form of $u$ and $v$ should be. As these forms are much too complicated to determine explicitly, we will not really use these explicit forms.

**Lemma 5.2.3.** *Let $u$, $v$, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, be as in the previous definition and suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are elements of $G_+(\Phi[[s,t]])$ and $G_-(\Phi[[s,t]])$, then $u$ must satisfy*

$$-2B(u_2) = -2s^2x_2 + s^3tT_x(y)_2 + 2s^4t^2Q_x(0, y_2)_2$$

*and*

$$-B(u_1 + 2u_2) = (\mathit{ad}\, v_1 - 1)2B(u_2) - sx_1 + s^2tQ_x(y)_1 + 2s^3t^2P_x(0, y_2)_1,$$

*where $P_x(y)$ denotes the additional Jordan-Kantor-like sequence pair operator. Since $B$ is invertible, a solution to the previous equations exists. Moreover, the quasi-inverses are necessarily unique.*

*Proof.* From $\exp(h)\exp(u)(\zeta) = \exp(v)^{-1}\exp(sx)\exp(ty)(\zeta)$, with $\zeta$ the grading element, one gets right away that the equalities of the lemma are necessary. Moreover, since we are working with formal power series and since $B = 1 + O(st)$, we conclude that $B$ is invertible with $B^{-1} = 1 + O(st)$. Additionally, since we are working with formal power series we can compute $u_1, v_1$ recursively by setting them to be $(u_1)_0 = 0$ and $(u_1)_1 = B^{-1}(sx_1 - s^2tQ_x(y) - 2s^3t^2P_x(0, y_2)_1)$ (analogous expressions for $v$) first and then iteratively computing what they should be. Specifically, one uses the following formula

$$(u_1)_n = (u_1)_{n-1} + B^{-1}(\mathrm{ad}\,((v_1)_{n-1} - (v_1)_n)2B(u_2)),$$

and a similar formula for $v$ (which depends on $u_{n-2}$ and $u_{n-1}$). By computing $v_2, u_2, v_3, u_3, \ldots$ we get two sequences. These sequences converge since the degree in $s$ and $t$ of what needs to be added will increase in every step. Lemma (5.1.2) applied to the projection idempotents $e_i : L \longrightarrow L_i$, shows that they are necessarily unique. ∎

**Lemma 5.2.4.** *The left and right quasi-inverse exist, and have the form of Lemma (5.2.3)*

*Proof.* We only need to prove existence, since the form of Lemma (5.2.3) is necessary. Lemma (2.4.13) shows us that the Lie-exponentials which have a compatibility with the grading are always of the form $\exp(\mathrm{ad}\, x)$. As such, it is sufficient to show that $u$ and $v$ are such Lie-exponentials.

We know that $\exp(ty)^{\exp(sx)^{-1}} = \sum s^it^j(i,j) = \prod \exp s^at^b[a,b]$, with the product increasing on $a/b \in \mathbb{Q}$ with $a, b$ coprime. As such, if the product of all exponentials $\exp s^at^b[a,b]$ with $a > b$ exists, then it is, necessarily $u \cdot (sx)^{-1}$. Similarly, the similar product with $a < b$ is $v$. Since all the exponentials $\exp[a,b]$ with $a \neq b$ are actual exponentials, we know that $u$ is given as

$$\exp(u) = \prod \exp(s^it^j[i,j]) \cdot \exp(sx)$$

where the product is over all the coprime $i > j$. As such $u$ and $v$ are Lie-exponentials, where the parts $U_i$ such that $u = 1 + U_1 + U_2 + \ldots$ can be defined from the parts of the product of these exponentials where the power of $t$ is $i$ lower than the power of $s$. Similar concerns show that $v$ is the left quasi-inverse. ∎

**Corollary 5.2.5.** *The TKK representatation satisfies the exponential property*

*Remark* 5.2.6. We could now reformulate, at least if $1/2 \in \Phi$, the exponential property using fixed $u$ and $v$ instead of the variable ones of definition (5.1.4). We refer to this property as the **fixed exponential property** and to its restricted variant as the **fixed restricted exponential property**. Specifically, one uses

$$\sum h'_{pq}s^pt^q = h' = \exp(v)^{-1}\exp(sx)\exp(ty)\exp(u)^{-1},$$

instead of $h$ in the usual definition of these properties, with $u, v$ as determined in this section.

The fixed and usual exponential properties are equivalent. The useful change is that the fixed restricted exponential property is better suited than the usual restricted exponential property for induction arguments.

## 5.3 The universal representation is 2-primitive

In this section, we replicate the arguments of Faulkner [Fau00, Section 6]. Despite some small changes to match the broader context we work in, these arguments remain mostly the same. We will be utilizing the theory developed in the previous sections. So, we assume that we are working with a sequence pair $G$ over a field $\Phi$ of characteristic different from 2 and 3. By Corollary (4.3.5), we know that $G$ has a defining Jordan-Kantor-like sequence pair representation. So, $G$ has a TKK representation.

Suppose that we have a sequence pair representation of $G$ in $A$. We define the following subalgebras of $A$: $\mathcal{X} = \langle x_i | x \in G_+(\Phi)\rangle$, $\mathcal{Y} = \langle y_i | y \in G_-(\Phi)\rangle$. Soon, we will define a third subalgebra $\mathcal{H}$. We will prove that $H = \mathcal{YHX}$, when $H$ is the universal representation. Additionally, we will prove that $H$ satisfies the (fixed) exponential property. We will use $h$, throughout this section, to denote

$$\sum h_{pq}s^p t^q = h = \exp(v)^{-1}\exp(sx)\exp(ty)\exp(u)^{-1},$$

for the fixed $u$ and $v$ as indicated in Remark (5.2.6). We will write $h(x, y)$ if we want to stress the dependence on $x$ and $y$. With this $h$, we define $\mathcal{H}$ as the subalgebra of $H$ generated by the $h_{pp}(x, y)$ for all $p \in \mathbb{N}, x \in G_\pm(\Phi), y \in G_\mp(\Phi)$.

We will indicate which lemma (or theorem) of [Fau00] the following lemmas generalize.

**Lemma 5.3.1 (Lemma 17).** *The fixed restricted exponential property $E_0$ holds for all sequence pair representations. Moreover, $h_{00}(x, y) = 1$ holds, for all $x \in G_\pm(\Phi), y \in G_\mp(\Phi)$.*

*Proof.* The moreover part is trivial. Notice that in $\exp(sx)\exp(ty)$ the 0-degree terms in $t$ form $\exp(sx)$. We note that the 0-degree part in $t$ of $u$ is $\exp(sx)$ and the 0-degree part in $t$ of $v$ is 1, so that the 0-degree part in $t$ of $\exp(v^{-1})\exp(sx)\exp(ty)\exp(u^{-1})$ is 1. The argument applies using the 0-degree part in $s$. So, $h_{ab}$ with $ab = 0$ is either 1 if $a = 0 = b$ or 0 if $a \neq 0$ or $b \neq 0$. ∎

**Lemma 5.3.2 (Lemma 18).** *Let $U$ be the universal sequence pair representation. For fixed $x \in G_\sigma(\Phi)$ and $y \in G_{-\sigma}(\Phi)$, the element $h$ is group like. Moreover, if only the fixed restricted exponential property $E_N$ holds, then $h_{ij}$, for $i \neq j$ with min$(i, j) \leq N + 1$, is primitive. Furthermore, if $E$ holds, then $\mathcal{H}$ is a Hopf subalgebra of $U$.*

*Proof.* Since $h$ is a product of exponentials of positive/negative homogeneous divided power series, which are all group like, it must be group like. We get

$$\Delta(h) = h \otimes h.$$

If we now compare the terms which belong to a fixed coefficient $s^i t^j$, with min$(i, j) \leq N + 1$, then we get

$$\Delta(h_{ij}) = \sum_{\substack{a+b=i \\ c+d=j}} h_{ac} \otimes h_{bd},$$

Assume now that the fixed restricted exponential property $E_N$ holds. So, the only terms which are non zero in the sum are the ones with $abcd = 0$ or $a = c$ and $b = d$. If we show that $h_{ab}$, with

$ab = 0$ equals 1 if $a = b = 0$ and 0 otherwise, then we will have shown that $h_{ij}$ is primitive if $i \neq j$. This is exactly Lemma (5.3.1). Furthermore, if $i = j$, then we get by the same observations that $(1, h_{11}, h_{22}, \ldots, h_{NN})$ forms a (finite) divided power series. So, if $E$ holds, then the $h_{ii}$ form an infinite power series. ■

We now define some functions on the monomials for general sequence pair representations. For a monomial $m = \prod_{i=1}^{k} (u_i)_{n_i}$ with $u_i \in G_{\pm}(\Phi)$ and $n_i \in \mathbb{N}$ and variable, but finite, $k$, we define the $\sigma$-**degree** as

$$\deg_\sigma m = \sum_{u_i \in G_\sigma} n_i.$$

Additionally, we define the **level** of $m$ by

$$\lambda(m) = \sum_{(i,j) \in L} n_i n_j,$$

where

$$L = \{(i,j) : i < j, \sigma_i = +, \sigma_j = -\},$$

where $\sigma_i = +$ if $x \in G_+(\Phi)$ and $\sigma_i = -$ if $x \in G_-(\Phi)$.

Something useful to note is, if $f(m_2) \leq f(m_2)'$ for $f = \deg_\pm$ and $\lambda$, then

$$\lambda(m_1 m_2 m_3) \leq \lambda(m_1 m_2' m_3), \tag{5.1}$$

holds, for all $m_1, m_3$, since $\lambda(m_1 m_2) = \lambda(m_1) + \lambda(m_2) + \deg_+(m_1)\deg_-(m_2)$ holds for all $m_1, m_2$.

**Definition 5.3.3.** Let $\mathcal{M}_{ab}(c)$ be the set of monomials $m$ with $\deg_+(m) \leq a, \deg_-(m) \leq b, \lambda(m) \leq c$.

**Lemma 5.3.4 (Lemma 19).** *If $\rho$ is a sequence pair representation of $G$, then*

$$h_{pq}(x,y) \equiv x_p y_q \mod \mathcal{M}_{pq}(pq-1).$$

*Proof.* Note that it is sufficient to prove that $x_p y_q$ is the only term in $h_{pq}$ with level $pq$ or greater, since the degrees of contributing monomials are already low enough, as $\deg_+$ should equal the degree of $s$ and $\deg_-$ the degree of $t$. Moreover, each term $v_i x_a y_b u_j$ in

$$h = \exp(v)^{-1} \exp(sx) \exp(ty) \exp(u)^{-1}$$

with $v_i u_j \neq 1$ will have, by definition, lower level. Hence, we are done. ■

**Lemma 5.3.5 (Lemma 20).** *If $\rho$ is a sequence pair representation of $G$, then, for $x, z \in G_+(\Phi)$, $y, w \in G_-(\Phi)$, the following elements are in $\mathcal{M}_{pq}(pq-1)$*

1. *$x_p y_q$ if $p \geq 2q$ or $2p \leq q$,*
2. *$x_a z_b y_q$ if $a + b = p \geq 2q$,*
3. *$x_p y_a w_b$ if $2p \leq a + b$.*

*Proof.* We see that the first kind of elements will be contained in $\mathcal{M}_{pq}(pq-1)$, as a consequence of the fact that the elements of the second or the third kind are contained in $\mathcal{M}_{pq}(pq-1)$. Note that, for each $u \in G_+(\Phi)$ and $v \in G_-(\Phi)$, by the definition of a sequence pair, each $(a, b)$ with $a \geq 2b$ lies in $\mathcal{M}_{ab}(ab-1)$.

As such, it is sufficient to see the second and third expression as linearizations of such $(a, b)$. Note that in $(a, b)_{x\cdot\lambda z, y}$ (where we stress the dependence of $(a, b)$ on $x\cdot\lambda z$ and $y$) the only terms which are not necessarily part of $\mathcal{M}_{ab}(ab-1)$ are $\sum_{i+j=n} \lambda^j x_i z_j y_b$. Since the whole should be in $\mathcal{M}_{ab}(ab-1)$ and since we can compare terms belonging to different powers of $\lambda$ we have proved that the second kind of elements is contained in $\mathcal{M}_{pq}(pq-1)$).

Similar considerations can prove the same for the third element, by making use of elements $(a, b)$ with the role of $+$ and $-$ interchanged. ■

We need the following fact about binomial coefficients.

**Lemma 5.3.6.** *If $d = \gcd\{\binom{n}{i} | 0 < i < n\}$, then*

$$d = \begin{cases} p & n = p^e, p \text{ prime} \\ 1 & \text{otherwise} \end{cases}.$$

*Proof.* This is exactly [Fau00, Lemma 21]. We also include the proof for completeness. Clearly, $d$ divides $n$. Suppose $p \mid d$ is a prime. Write $n = p^e m$ with $p \nmid m$. In $\mathbb{Z}_p[t]$ we have

$$1 + t^n = (1+t)^n = (1 + t^{p^e})^m = 1 + mt^n + \ldots$$

So, $p \mid m$ if $m \neq 1$, which is a contradiction. So, suppose now that $d = p^f$ and $n = p^e$. If $f > 1$, then is

$$(1+a)^n \equiv 1 + a^n \mod p^2,$$

So, $a^n \equiv a \mod p^2$ by induction on $a$, as $1^n = (1+0)^n \equiv 1 + 0^n = 1$. However, $p^n \equiv 0 \mod p^2$, but $p$ is not. So, $f = 1$. ■

**Lemma 5.3.7 (Lemma 22).** *Suppose that the fixed restricted exponential property $E_N$ holds for a sequence pair representation $\rho$ of $G$. Consider $x \in G_\pm(\Phi)$ and $y \in G_\pm(\Phi)$. Take $p \neq q$ with $\min(p, q) \leq N+1$, then $x_p y_q \in \mathcal{M}_{pq}(pq-1)$.*

*Proof.* Suppose, first, that $\min(p, q) \leq N$. Now, consider that

$$\exp(sx)\exp(ty) = \exp(v) h \exp(u).$$

This implies $x_p y_q = \sum v_i h_{kk} u_j$ where $v_i$ and $u_j$ are terms depending only on $u$ and $v$ and degrees in $s$ and $t$. Lemma (5.3.4) shows that $h_{kk} \in \mathcal{M}_{kk}(k^2) \subset \mathcal{M}_{k^2}(pq-1)$, as $k^2 \leq \min(p, q)^2 < pq$ indicates. So, if $v_i, u_j \neq 0$ we see that $v_i h_{kk} u_j \in \mathcal{M}_{pq}(k^2) \subset \mathcal{M}_{pq}(pq-1)$. Therefore, we see that

$$x_p y_q \in \mathcal{M}_{pq}(pq-1).$$

Now, we prove the lemma for $p, q$ such that $\min(p, q) = N+1$. Since we assume that $1/2 \in \Phi$, we can easily compute $x \cdot \lambda x$ for $x$ an element of a sequence $\Phi$-group $G$. Namely, there is a unique way to write $x = ah$ with $(-a)a = 1$ and $h \in H^1(\Phi)$. So, $x \cdot \lambda x = (1+\lambda)a \cdot h'$ with $h' = (1+\lambda^2)h$, where we used the module structure of $H^1$ instead of the usual scalar multiplication on the group. Assume without loss of generality that $p = N+1$. We see that

$$((1+\lambda)\cdot x)_p y_q = \sum_{i+2j=p} (1+\lambda)^i a_i (1+\lambda^2)^j h_{2j} y_q.$$

Hence we see that this equals, with $p_{even} = 1$ if $p$ is even and $0$ otherwise,

$$(1+\lambda)^p a_p y_q + p_{even}(1+\lambda^2)^{(p/2)} h_p y_q \mod \mathcal{M}_{pq}(pq-1),$$

since $h_i y_q$ for $i < p$ is contained, by the first part, in $\mathcal{M}_{iq}(iq-1)$. We see that the term belonging to $\lambda^i$, $i \neq 0, p$, in $(1+\lambda)^p a_p y_q$ is $\binom{p}{i} a_p y_q$, this should, on the other hand, equal to $a_{p-i} a_i y_q \in \mathcal{M}_{pq}(pq-1)$, as $a \cdot \lambda a = (1+\lambda) \cdot a$. So, we see that $\binom{p}{i} a_p y_q \in \mathcal{M}_{pq}(pq-1)$ for all $i \neq 0, p$. If $p$ is even, we can do exactly the same for $h_p y_q$ to get that $\binom{p/2}{i} h_p y_q \in \mathcal{M}_{pq}(pq-1)$ for all $0 < i < p/2$.

Now, we do nearly the same thing with $y$ to get a different set of binomial coefficients, so that we can assume that the greatest common divisor of all these coefficients is not a power of a prime bigger than 3.

We can assume that $q < 2p$, by Lemma (5.3.5). We know that $i + j = q$ implies that either $i < p$ or $j < p$. So, of $i$ and $j$ at least one is smaller than or equal to $N$. We can do nearly same thing as we did before, to prove that $\binom{q}{l} a_p y_q \in \mathcal{M}_{pq}(pq-1)$. We write $y = bg$, so that $(-b)b = 1$ and $g \in H^1_-(\Phi)$. Since $bg = gb$ and since we are working with a sequence $\Phi$-group representation, we know that $[b_i, g_j] = 0$ for all $i$ and $j$, by comparing the coefficients of $\lambda$ in $[\lambda \cdot g, b]_{i+j} = 0$. As such we can guarantee in terms $a_p b_i g_j$ that $i \leq N$ or we replace it with the term $a_p g_j b_i$ with $j \leq N$. We conclude that

$$a_p((1+\mu)y)_q = a_p(1+\mu)^q b_q + q_{even}(1+\mu^2)^{q/2} a_p g_q \mod \mathcal{M}_{pq}(pq-1).$$

So, by comparing terms belonging to $\mu$ we get $\binom{q}{l} a_p b_q \in \mathcal{M}_{pq}(pq-1)$ for all $0 < l < q$. We can do the same for $g$ to get that all $\binom{q/2}{i}$ for $0 < i < q/2$ are part of $\mathcal{M}_{pq}(pq-1)$.

Thus, we know that

$$x_p y_q = a_p b_q + p_{even} h_p b_q + q_{even} a_p g_q + p_{even} q_{even} h_p g_q \mod \mathcal{M}_{pq}(pq-1)$$

and that all $\mathbb{Z}-$multiples with coefficients $\binom{p}{i}, \binom{q}{j}$ (or $\binom{p,q/2}{i}$ for terms with $h$ and $g$ instead of $a$ and $b$) for $i \neq p, 0, j \neq q, 0$ of these four terms of the right hand side are contained in $\mathcal{M}_{pq}(pq-1)$. If we can show that the greatest common divisor of all those binomial coefficients is a power of 1, 2 or 3 then we know that the terms without the binomial coefficients are contained in $\mathcal{M}_{pq}(pq-1)$. This is the case since we assumed that $p < 2q$ and $2p > q$. Specifically, in all cases of binomial coefficients $\binom{a}{i}, \binom{b}{j}$, we know that $a < 4b$ and $b < 4a$. As such, if $a$ and $b$ are powers of the same prime, they can only be powers of 2 and 3. These primes are assumed to be invertible. ■

*Remark* 5.3.8. This is the only lemma in this section where we use the fact that we can divide by 3 (modulo the fact that we use that the TKK representation satisfies the exponential property). This can easily be remedied by considering Jordan-Kantor-like sequence pair representations instead of just sequence pair representations. First we need to generalize Lemma (5.3.5) so that it includes $x_p h_q \in \mathcal{M}(pq-1)$ if $h \in H^1_-(\Phi)$ and $2p > 3q$ and $h'_p y_q \in \mathcal{M}(pq-1)$ if $h \in H^1_+(\Phi)$ and $3p < 2q$. This lets us use a better bound on the greatest common divisor of the multiples of a term belonging in $\mathcal{M}_{pq}(pq-1)$, so that 3 cannot be that divisor. Notwithstanding the previous observations, we do not consider characteristic 3 in this section, as we do not know whether the exponential property holds for the TKK representation.

**Lemma 5.3.9 (Lemma 23).** *If a sequence pair representation $\rho$ of $G$ satisfies the fixed exponential property, then*

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \mathcal{Y}\mathcal{H}\mathcal{X}.$$

*If it satisfies the fixed restricted exponential property $E_N$, then, for $r, s$ with min$(r, s) \leq N + 1$, $x \in G_+(\Phi)$, $y \in G_-(\Phi)$, we have that $h_{rs} \in \mathcal{Y}\mathcal{H}\mathcal{X}$.*

*Proof.* We show by induction on $n$ that if $E_N$ holds and $\min(r,s) \le N+1$, then

$$\mathcal{M}_{rs}(n) \subset \mathcal{YHX}.$$

Clearly, $\mathcal{M}_{rs}(0)$ is a subset of $\mathcal{YX}$. So, let $m \in \mathcal{M}_{rs}(n)$ be a monomial such that $m \notin \mathcal{M}_{rs}(n-1)$. If $m$ factors as $m_1 x_p y_q m_3$ with $pq \neq 0$, $x \in G_+(\Phi)$, $y \in G_-(\Phi)$, then Lemma (5.3.7) shows that $p$ should equal $q$. Moreover, $m$ cannot factor as $m_1 x_{p_1} z_{p_2} y_q m_3$, with $x$ and $y$ as before and $z \in G_+(\Phi)$ with $p_1 p_2 q \neq 0$, Specifically, we know that $p_2 = q$ by the previous observation. If we use equation (2.1) and the fact that $H^1_\sigma(\Phi)_i$ commutes with all $G_\sigma(\Phi)_j$ to rewrite

$$x_{p_1} z_{p_2} = \sum_{\substack{a+c=p_1\\ b+c=p_2}} z_b x_a [x,z]_{2c} = \sum_{\substack{a+c=p_1\\ b+c=p_2}} [x,z]_{2c} z_b x_a,$$

then we get that the only terms such that

$$[x,z]_{2c} z_b x_a y_q = z_b x_a [x,z]_{2c} y_q \notin \mathcal{M}_{(p_1+p_2)q}((p_1+p_2)q-1)$$

are the ones with $a = q, 0$ and $2c = q, 0$. Thus, $p_1 \ge q$ and $p_1 + p_2 \ge 2q$ hold, which implies, using Lemma (5.3.5), that these terms are contained in $\mathcal{M}_{(p_1+p_2)q}((p_1+p_2)q-1)$. Analogously one can show that factorizations of the form $x_p y_{q_1} w_{q_2}$ with $w \in G_-(\Phi)$, are also impossible.

Therefore, we get $m = m_1 \prod((x_i)_{p_i}(y_i)_{p_i}) m_3$ with $x_i \in G_+(\Phi)$, $y_i \in G_-(\Phi)$, $m_1 \in \mathcal{Y}$ and $m_3 \in \mathcal{X}$. This means, by Lemma (5.3.4), that

$$m \equiv m_1 (\prod h_{p_i p_i}(x_i, y_i)) m_3 \mod \mathcal{M}_{rs}(n-1),$$

where we wrote $h_{pq}(a,b)$ to stress the dependence of $h_{pq}$ on $a$ and $b$. From the induction hypothesis, we conclude $m \in \mathcal{YHX}$. Hence, the fixed exponential property implies $\langle \mathcal{X}, \mathcal{Y} \rangle = \mathcal{YHX}$.

Furthermore, Lemma (5.3.4) lets us conclude that $h_{rs}(x,y) \in \mathcal{M}_{rs}(rs) \subset \mathcal{YHX}$. ■

Now, we take a well-ordered basis $u_i$, for $i \in I$ of $G_+(\Phi)$, which corresponds to the vector space $G/H^1_+(\Phi) \oplus H^1_+(\Phi)$. We assume that this basis has a partition as a basis of $G/H^1_+(\Phi)$ and one of $H^1_+(\Phi)$ where we let $u_i$, a basis vector for $G/H^1_+(\Phi)$, correspond to the unique $u_i \in G_+(\Phi)$ with $(-u_i)u_i = 1$. We assume that we have a similar basis $v_j, j \in J$ of $G_-(\Phi)$. Let $u_i^*, v_j^*$ be their dual bases. In what comes, we shall denote with $k$ a map $I \longrightarrow \mathbb{N}$ with only a finite amount of $x \in I$ which have non zero image. We let this $k$ correspond to

$$m_k^+ = \prod g_i^{(k(i))} \prod h_i^{(2k(i))}$$

with the order of multiplication that corresponds to the order of the basis, and where we wrote $g_i^{k(i)}$ instead of $(g_i)_{k(i)}$ to represent the $k(i)$th term of the sequence $g_i$. Additionally, we split the product, according to being part of $G/H^1_+$ (namely the $g$'s) and being part of $H^1$ (namely the $h$'s). We shall use $l$ for similar functions $J \longrightarrow \mathbb{N}$. In addition, for $b \in \mathcal{H}$ we define $m_{kl}(b) = m_l^- b m_k^+$.

We recall for Jordan-Kantor-like sequence pairs, that there is the TKK representation in the endomorphism algebra of the Lie algebra $H^1_- \oplus G_-/H^1_- \oplus \mathbf{InDer}(G) \oplus G_+/H^1_+ \oplus H^1_+$. We extend the maps $u_i^*, v_j^*$ of the dual basis to the whole of the TKK Lie algebra.

We define maps

$$\lambda_i^+(a) = f_i u_i^*(S(a) \cdot \zeta),$$

with $\zeta$ the grading element of the TKK Lie algebra, and $\cdot$ corresponding to the action of the universal representation on the TKK Lie algebra and $f_i = 1$ for $i$ corresponding to $G/H^1$ and $1/2$ for $H^1$. For the $v_i$ we set

$$\lambda_i^-(a) = f_i v_i^*(a \cdot \zeta).$$

**Lemma 5.3.10 (Lemma 24).** *The maps $\lambda_i^+$ satisfy*

$$\lambda_i^+(bx_1) = \epsilon(b)u_i^*(x),$$

$$\lambda_i^+(\mathcal{YH}) = 0,$$

$$\lambda_i^+(bh_2) = \epsilon(b)u_i^*(h),$$

*for $x \in G_+(\Phi)$, $h \in H_+^1(\Phi)$ and $b \in \mathcal{Y}$.*

*Proof.* First, we determine the action of elements in $\mathcal{H}$ on $\zeta$. We note that $B(x,y) = 1 + st[1,1] + s^2t^2[2,2] + \dots$ acts as an automorphism of the TKK Lie algebra and that $\epsilon[i,i] = \delta_{i0}$. Moreover, $S(B(x,y)) = B(x,y)^{-1}$.

Now we are ready to compute the action of $B(x,y)$ on $\zeta$. We compute, using the fact that it acts as an automorphism, that

$$B(x,y) \cdot [\zeta, u] = [B(x,y) \cdot \zeta, B(x,y) \cdot u],$$

which implies that

$$(\text{ad } \zeta)^{B(x,y)} = \text{ad } (B(x,y) \cdot \zeta).$$

Notice that $(\text{ad } \zeta)^{B(x,y)}(u) = \sigma(u)u$, for $u \in L_{\sigma(u)}$. As such, we can conclude that $B(x,y)(\zeta) = \zeta$, as the equality between elements of $\mathbf{InDer}(G)$ is determined by their action on the rest of the Lie algebra. Consider that $\mathcal{H}$ is generated by the $h_{ii}$, and that $h(x,y) = B(sx, ty)$ in the TKK representation. Moreover, $h(x,y)$ is group-like. So, we get $\epsilon(h(x,y)) = 1$ and $S(h(x,y)) = h(x,y)^{-1}$. So, we obtain that $h(x,y) \cdot \zeta = \epsilon(h(x,y))\zeta = S(h(x,y)) \cdot \zeta$. So, we conclude that $u \cdot \zeta = \epsilon(u) \cdot \zeta = S(u) \cdot \zeta$ for all $u \in \mathcal{H}$, as $\mathcal{H}$ is generated by the $h_{ii}$.

The first and third equation immediately follow from straightforward computation. We execute the computation for the first equation (but for the third equation one just needs to substitute a 2 for each 1), we compute

$$\begin{aligned} \lambda_i^+(bx_1) &= u_i^*((x_1^{-1})S(b) \cdot \zeta) \\ &= u_i^*(-\text{ad } x_1(\epsilon(b)\zeta)) \\ &= \epsilon(b)u_i^*(x), \end{aligned}$$

where we needed that $x_1^{-1} = -x_1$ for all $x$. The second equation follows from the fact that

$$u_i^*(G_- \oplus \text{InDer}(G)) = 0.$$

This finishes the proof. ∎

*Remark* 5.3.11. There are similar expressions for $\lambda_i^-$, but we will not prove them.

Now we will recursively construct some functions which will allow us to show that each element of $\mathcal{YHX}$ can be written, in a certain sense, in a unique way. Recall that $k$ represents a function $I \longrightarrow \mathbb{N}$ with finite support. We can define sums of such functions. Furthermore, set $\mid k \mid = \sum k(i)$, and let $s = s(k)$ be the minimal element of $I$ with $k_s \neq 0$. Additionally, we define $\bar{k}$ to be $k - \delta_{i,s(k)}$ where $\delta$ represents the krönecker delta. We can do the same for functions $l : J \to \mathbb{N}$.

Note that for the bases $u_i, v_i$, we have

$$\Delta(u_i^{(k(i))}) = \sum_{k' + k'' = k(i)} u_i^{(k')} \otimes u_i^{(k'')}$$

As such, we get the following expression for monomials $m_k$

$$\Delta(m_k) = \sum_{k'+k''} m_{k'} \otimes m_{k''},$$

where we now sum over functions $k : I \longrightarrow \mathbb{N}$. The most general form we need is

$$\Delta(m_{kl}(b)) = \sum m_{k'l'}(b'_t) \otimes m_{k''l''}(b''_t),$$

where we sum over $k' + k'' = k, l' + l'' = l$ and where

$$\Delta(b) = \sum_i b'_i \otimes b''_i.$$

We recursively define $\beta_{pq} : U \longrightarrow U$, on the universal representation $U$, as

$$\beta_{pq} = \begin{cases} \mathrm{Id} & \text{if } p = q = 0 \\ (\lambda^-_{s(q)} \otimes \beta_{0\bar{q}}) \circ \Delta & \text{if } p = 0, q \neq 0 \\ (\lambda^+_{s(p)} \otimes \beta_{\bar{p}q}) \circ \Delta & \text{if } p \neq q \end{cases}.$$

We recall and stress that $p(i)$, for $i$ related to $H^1$, corresponds $u_i^{2p(i)}$, instead of $u_i^{p(i)}$. This explains why we do not need to differentiate in the definition of $\beta$ between the basis elements.

**Lemma 5.3.12 (Lemma 25).** *Let $\gamma : G \longrightarrow U$ be the universal representation. If $|k| \leq |p|$ and $|l| \leq |q|$, then*

$$\beta_{pq} \circ m_{kl} = \begin{cases} 0 & \textit{if } (p,q) \neq (k,l) \\ \mathit{Id} & \textit{if } (p,q) = (k,l) \end{cases}.$$

*Proof.* The base case $p = q = 0$ is trivial. We prove that if $p \neq 0$ and if the lemma holds for all $k$ with $|k| < |p|$, then it holds for $p$ as well. The case $p = 0$ is analogous to what we will prove, so we assume this to be proven.

Take $b \in \mathcal{H}$. We have

$$\beta_{pq}(m_{kl}(b)) = \sum \lambda^+_{s(p)}(m_{k'l'}(b'_t))\beta_{\bar{p}q}(m_{k''l''}(b''_t)), \tag{5.2}$$

where we sum over $k' + k'' = k, l' + l'' = l$ and $\Delta(b) = \sum_t b'_t \otimes b''_t$. Clearly, $k' + k'' = k$ implies that $|k''| \leq |k| \leq |p|$. If $|k''| = |p|$, then $k' = 0$ must hold, and

$$\lambda^+_{s(p)}(m_{k'l'}(b'_t)) \in \lambda^+_{s(p)}(\mathcal{YH}) = 0.$$

We, thus, assume that $|k''| \leq |p| - 1 = |\bar{p}|$. So, we can apply the induction hypothesis, which yields

$$\beta_{\bar{p}q}(m_{k''l''}(b''_t)) = \begin{cases} 0 & \text{if } (\bar{p},q) \neq (k'',l'') \\ b''_t & \text{if } (\bar{p},q) = (k'',l'') \end{cases}.$$

Moreover, if $(\bar{p}, q) = (k'', l'')$ holds, then $|l| \leq |q|$ implies that $l = q$ and we obtain $l' = 0$. Furthermore, $|k'| + |\bar{p}| = |k| \leq |p|$ holds, so we see that $|k'| \leq 1$. Thus, either $k' = 0$ holds or we know that $k' : i \to \delta_{ei}$ for some $e$. In the first case, we get $\lambda^+_{s(p)}(m_{00}(b'_t)) = \lambda^+_{s(p)}(b'_t) = 0$, while we obtain in the second case that

$$\lambda^+_{s(p)}(m_{k'l'}(b'_t)) = \lambda^+_{s(p)}(b'_t u_e^{(1)}) = \epsilon(b'_t)u^*_{s(p)}(u_e).$$

Now, we see that all terms are 0 in equation (5.2) except those with $\bar{p} = k''$, $q = l = l''$ and $e = s(p)$. Moreover, if this holds, then $k = \bar{p} + k' = p$. Thus, $\beta_{pq}(m_{kl}(b)) = 0$ holds if $(p, q) \neq (k, l)$. Lastly, if $(p, q) = (k, l)$, then we get

$$\begin{aligned}\beta_{pq}(m_{pq}(b)) &= \sum_t \epsilon(b'_t)b''_t \\ &= (\epsilon \otimes \mathrm{Id})(\Delta(b)) \\ &= b.\end{aligned}$$

■

**Lemma 5.3.13 (Lemma 26).** *Let $\gamma : G \longrightarrow U$ be the universal representation. Every $a \in \mathcal{YHX}$ can be written uniquely as*

$$a = \sum m_{kl}(c_{kl}), \quad \textit{with } c_{kl} \in \mathcal{H}.$$

*Also, if $a \in \mathcal{YHX}$ is primitive, then $a \in (H^1_-)_2 \oplus (G_-)_1 \oplus \mathcal{H} \oplus (G_+)_1 \oplus (H^1_+)_2$, where $(G_+)_1$ denotes the first coordinate of each sequence corresponding to $g \in G_+(\Phi)$ (and the other expressions have similar meanings).*

*Proof.* Equation (2.2) shows that we can assume that each $x \in \mathcal{X}$ is a sum of products $\prod g_i^{(k_i)}$, weakly increasing on the $i$. Equation (2.1) shows that we can rewrite these to be products which are strictly increasing on the $i$ (using the fact that all $h_k$ for $h \in H^1_\sigma(\Phi)$ commute with every $g_i$ for $g \in G_\sigma(\Phi)$). As such, each $a$ is of the form

$$a = \sum m_{kl}(c_{kl}),$$

for some $c_{kl} \in \mathcal{H}$. So, we want to show uniqueness. To do that, it is sufficient to prove that $a = 0$ implies that $c_{kl} = 0$ for all $k$ and $l$. We show both statements of the lemma at the same time; we will show that the $c_{kl}$ are unique for primitive elements, including 0, in such a way that it clearly shows that the possible primitive elements are the ones mentioned in the lemma.

We can suppose that each $a$ and each of the terms $m_{kl}(c_{kl})$ contributing to $a$ belong to the same grading component $U_d$ in the $\mathbb{Z}$-grading on $U$. If there is some $c_{kl} \neq 0$, choose $p, q$ with $c_{pq} \neq 0$ and $|p|$ maximal. Hence, $c_{kl} \neq 0$ indicates that $|k| \leq |p|$. Furthermore, as $\mathcal{H} \subset U_0$, we obtain

$$d = |k| - |l| = |p| - |q|,$$

so $|l| \leq |q|$ must hold as well. Lemma (5.3.12) proves that $c_{pq} = \beta_{pq}(a)$ holds. In particular, $a = 0$ implies that all $c_{kl}$ must be 0.

Now, we consider the three cases in the definition of $\beta_{pq}$. If $p = q = 0$, then $a = \beta_{00}(a) = c_{00} \in \mathcal{H}$. For the remaining two cases, we shall show that $m_{pq}(c_{pq})$ is a primitive element of the right form, and use induction on the number of remaining terms in $a - m_{pq}(c_{pq})$. If $p = 0, q \neq 0$, then we get

$$\begin{aligned}c_{0q} &= \beta_{0q}(a) \\ &= (\lambda^-_{s(q)} \otimes \beta_{0\bar{q}})(a \otimes 1 + 1 \otimes a) \\ &= \lambda^-_{s(q)}(a)\beta_{0\bar{q}}(1) \\ &= \begin{cases} 0 & \text{if } \bar{q} \neq 0 \\ \lambda^-_{s(q)}(a) & \text{if } \bar{q} = 0 \end{cases},\end{aligned}$$

where the second equality holds because of the grading. Since $c_{pq} \neq 0$, we conclude that $p = 0, q \neq 0$ implies that $\bar{q} = 0$ and $m_{0q}(c_{0q}) = \lambda^-_{s(q)}(a)v^{(i)}_{s(q)}$ with $i = 1, 2$ depending on whether $s(q)$ corresponds to a basis vector of $G/H^1_-$ or $H^1_-$.

Similarly, if $p \neq 0$, then we get

$$\begin{aligned} c_{pq} &= \beta_{pq}(a) \\ &= (\lambda^+_{s(p)} \otimes \beta_{\bar{p}q})(a \otimes 1 + 1 \otimes a) \\ &= \lambda^+_{s(p)}(a)\beta_{\bar{p}q}(1) \\ &= \begin{cases} 0 & \text{if } (\bar{p}, q) \neq (0,0) \\ \lambda^+_{s(p)}(a) & \text{if } (\bar{p}, q) = (0,0) \end{cases}. \end{aligned}$$

So, we conclude that $\bar{p} = 0$ and $m_{pq}(c_{pq}) = \lambda^+_{s(p)}(a)u^{(i)}_{s(p)} \in (G_+)_1 \oplus (H^1_+)_2$. ∎

**Theorem 5.3.14 (Theorem 27).** *Suppose that $\rho : G \longrightarrow A$ is a sequence pair representation over $\Phi$. This representation satisfies the fixed exponential property and $\langle \mathcal{X}, \mathcal{Y} \rangle = \mathcal{YHX}$.*

*Proof.* By Lemma (5.3.9), it suffices to prove the fixed exponential property. We prove this property for the universal representation. Therefrom, it follows for every sequence pair representation. We show that the fixed exponential property holds, using induction. Namely, we show the fixed restricted exponential property $E_N$ for each $N \in \mathbb{N}$. We know that $E_0$ holds by Lemma (5.3.1).

Suppose that $E_N$ holds and let $p \neq q$ with $\min(p, q) \leq N + 1$. Consider $x \in G_+(\Phi), y \in G_-(\Phi)$. From Lemmas (5.3.9) and (5.3.2), it follows that $h_{pq} \in \mathcal{YHX}$ and that it is a primitive element. Lemma (5.3.13) lets us conclude that $h_{pq} \in (H^1_\pm)_2 \oplus (G_\pm)_1$.

We know that the TKK representation, which we see as a morphism $\xi$ from the universal representation, satisfies the fixed exponential property since $1/6 \in \Phi$. So, $\xi(h_{pq}) = 0$ is true. However, the TKK representation is faithful. Hence, we conclude that $h_{pq} = 0$ in the universal representation. This shows that $E_{N+1}$ holds. Therefore, the universal representation satisfies the fixed exponential property. ∎

**Corollary 5.3.15 (Corollary 28).** *Let $\gamma : G \longrightarrow U$ be the universal representation, then the primitive elements are determined as*

$$\mathcal{P}(U) = (H^1_-)_2 \oplus (G_-)_1 \oplus \mathcal{P}(\mathcal{H}) \oplus (G_+)_1 \oplus (H^1_+)_2,$$

*where $\mathcal{P}(A)$ stands for the primitive elements of $A$. Moreover, each $h \in U$ can be expressed uniquely as a sum of monomials $m_{pq}(b_k)$.*

**Corollary 5.3.16.** *Each sequence pair representation of $G$ is a Jordan-Kantor-like sequence pair representation.*

*Proof.* By Corollary (5.3.15) we know what the primitives are of the universal representation. Utilizing that, it is easy to see that the universal representation is a Jordan-Kantor-like sequence pair representation. Hence, each sequence pair representation is a Jordan-Kantor-like sequence pair representation. ∎

*Remark* 5.3.17. Notice how easily we can construct from the previous Hopf algebra $U$ all other ingredients in the construction. Firstly, we can recover from $U$ the sequence pair. Secondly, we can recover the Jordan-Kantor pair. Additionally, we can consider the subalgebra $\tilde{L}$ of $\mathcal{P}(U)$ generated by $(H^1_-)_2 \oplus (G_-)_1 \oplus (G_+)_1 \oplus (H^1_+)_2$, which coincides with the universal central extension of the TKK Lie algebra $L$ associated to the Jordan-Kantor pair. Note that $\tilde{L}$ equals $\mathcal{P}(U)$ except maybe on the 0-graded part. Specifically, we can construct $L$ from $\tilde{L}$ by taking the quotient with respect to Ker(ad$^\sharp$), with

$$\mathrm{ad}^\sharp : \tilde{L} \longrightarrow \mathrm{End}_\Phi(L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2),$$

by mapping $l$ to the restriction of ad $l$.

# 6 The universal enveloping algebra

We assume, in this chapter, that $\Phi$ is a field of characteristic $0$. We will show that for (Jordan-Kantor-like) sequence pairs over $\Phi$, the universal enveloping algebra of the TKK Lie algebra is isomorphic to the universal representation. We could try to follow [Fau00] by explicitly checking that the sequence pair representation in question is a sequence pair representation. However, this would be quite challenging. It is easier to directly use the Hopf algebra structure of the enveloping algebra and the fact that in characteristic $0$ the primitive elements are exactly those elements that correspond to elements of the underlying Lie algebra.

## 6.1 The universal enveloping algebra

We introduce the universal enveloping algebra roughly following Hall [Hal15].

**Definition 6.1.1.** Suppose that $L$ is a Lie algebra and $A$ is an associative algebra. We say that a linear map $\phi : L \longrightarrow A$ is a **representation** if $\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$.

**Definition 6.1.2.** For a Lie algebra $L$, we define the **Tensor algebra** $T(L)$. This algebra is defined as

$$T(L) = \bigoplus_{k=0}^{\infty} L^{\otimes k}.$$

This is an associative unital algebra with multiplication defined by

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = (v_1 \otimes \cdots v_n \otimes w_1 \otimes \cdots \otimes w_m).$$

There is an inclusion $i$ of $L$ into $T(L)$. Note that for each linear map

$$\phi : L \longrightarrow A$$

into an associative unital algebra $A$, there exists a unique algebra morphism

$$\psi : T(L) \longrightarrow A$$

such that $\psi(i(x)) = \phi(x)$. To be specific, the map is determined by

$$\psi(x_1 \otimes \cdots \otimes x_n) = \phi(x_1) \cdots \phi(x_n).$$

Now we try to find a minimal two-sided ideal $I$ of $T(L)$ such that for $U = T(L)/I$ the inclusion of $L \longrightarrow U$ is actually a Lie algebra representation. We take the ideal $I$ generated by $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in L$. Note that, on the other hand, for each representation $\phi : L \longrightarrow A$ the unique $\psi : T(L) \longrightarrow A$ factors through $U$. So, we conclude that $U$ is a representation satisfying a certain universal property. Namely, for each representation $\phi : L \longrightarrow A$ there exists a unique $\psi : U \longrightarrow A$ such that $\phi = \psi \circ i$.

*Remark* 6.1.3. Note that if $L$ is a graded Lie algebra, then $U$ inherits that grading. Specifically, on $T(L)$ there is a natural grading, by setting $L_{a_1} \otimes \ldots \otimes L_{a_n}$ to be $(\sum_{i=1}^n a_i)$-graded. Note that the ideal $I$ is compatible with the grading.

**Definition 6.1.4.** Let $L$ be a Lie algebra. An algebra $A$ with linear map $i : L \longrightarrow A$ such that for all representations $\phi : L \longrightarrow B$ there exists a unique $\psi : A \longrightarrow B$ such that $\phi = \psi \circ i$ is called **the universal enveloping algebra**. The 'the' is justified as this algebra is unique up to isomorphism. Note that the previously constructed $U$ is the universal enveloping algebra of the Lie algebra $L$.

We can endow $U$ with a Hopf algebra structure. We define some operators on $T$ and check if these make $U$ into a Hopf algebra, by checking whether these operators make $I$ a Hopf ideal. We define $\Delta : T(L) \longrightarrow T(L) \otimes T(L)$ by

$$\Delta(1) = 1 \otimes 1$$

and

$$\Delta(i(x)) = i(x) \otimes 1 + 1 \otimes i(x),$$

for $x \in L$, note that $\Delta(x_1 \otimes \cdots \otimes x_n) = \Delta(x_1) \cdots \Delta(x_n)$ yields the definition on the whole of $T(L)$. One easily checks[1] that $x \otimes y - y \otimes x - [x, y]$ is primitive. We conclude that $\Delta(I) \subset I \otimes T(L) + T(L) \otimes I$. Note that $\Delta$ is coassociative. The counit is easy to define, namely $\epsilon(k) = k$ for $k \in L^{\otimes 0}$ and $\epsilon(i(x)) = 0$ for $x \in L$. Since it acts as a counit on the generators $i(x)$ and 1, namely

$$(\epsilon \otimes \mathrm{Id})\Delta(i(x)) = i(x) = (\mathrm{Id} \otimes \epsilon)\Delta(i(x)),$$

we see that it actually is a counit. Note that $\epsilon(I) = 0$. The antipode is, easily, defined by $S(i(x)) = -i(x)$ and

$$S(u \otimes \cdots \otimes v) = S(v) \otimes \cdots \otimes S(u),$$

as it needs to be an algebra anti-morphism. That this is an antipode follows from the fact that

$$\mu \circ (\mathrm{Id} \otimes S)\Delta(i(x)) = \eta(\epsilon(i(x))) = 0 = \mu \circ (S \otimes \mathrm{Id})\Delta(i(x)),$$

for generators $i(x)$. So, we see that $U$ is a Hopf algebra with $\Delta, \epsilon, S$ induced by the ones defined on $T(L)$ as $I$ is a Hopf ideal.

**Theorem 6.1.5 (Poincarré-Birkhoff-Witt).** *Suppose that $L$ is a finite dimensional Lie algebra with basis $X_1, \ldots, X_k$. The elements of the form*

$$i(X_1)^{n_1} \ldots i(X_n)^{n_n},$$

*with each $n_i$ a non-negative integer, span $U$ and are linearly independent. In particular, $i : L \longrightarrow U$ is injective.*

*Proof.* See [Hal15, Theorem 9.9]. ■

**Proposition 6.1.6.** *Let $L$ be a finite dimensional Lie algebra over $\Phi$. The primitive elements of $U$ are, exactly, the elements $i(x), x \in L$.*

*Proof.* We note that

$$\mu \circ \Delta(x) = 2x,$$

for primitive $x$. By the Poincarré-Birkhoff-Witt theorem, we know that we can write each $x$ uniquely as

$$\sum_{\mathbf{n}} \lambda_{\mathbf{n}} \prod_{i=1}^{n} i(X_i)^{\mathbf{n}_i},$$

[1] This is in some sense "the same" computation that shows that $[\mathrm{ad}\, x, \mathrm{ad}\, y] = \mathrm{ad}\, [x, y]$.

with $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_n)$ and $n$ the dimension of $L$. What we want to prove is that $\mu \circ \Delta(x) = 2x$ implies that all $\mathbf{n}$ satisfy

$$|\mathbf{n}| = \sum \mathbf{n}_i = 1.$$

This is the case, since

$$\Delta \circ \mu(\prod_{i=1}^{n} i(X_i)^{\mathbf{n}_i}) = 2^{|\mathbf{n}|} \prod_{i=1}^{n} i(X_i)^{\mathbf{n}_i},$$

and since we are working in characteristic 0, so that $2^p = 2$ has only 1 solution, namely $p = 1$. ∎

*Remark* 6.1.7. One can even prove something stronger than the previous proposition. Milnor and Moore ([MM65, Theorem (5.18)]) proved that for each Lie (super-)algebra the functors $U$ (mapping a Lie super-algebra to its universal enveloping algebra) and $P$ (mapping a Hopf algebra to the Lie algebra of its primitive elements) satisfy $PU = \mathrm{Id}$ and $UP = \mathrm{Id}$, at least when we look at Lie (super-)algebras and certain cocommutative Hopf algebras generated by the primitive elements over fields $\Phi$ with characteristic 0. What the exact Hopf algebras are, does not matter. What is interesting, is that $PU = \mathrm{Id}$ on the Lie algebras over fields with characteristic 0.

## 6.2 An isomorphism in characteristic $0$

Consider a Jordan-Kantor pair $P$ with TKK Lie algebra $L$ (using the inner structure algebra with grading element) and its universal central extension $\tilde{L}$. An exact description of $\tilde{L}$ has been given by Benkart and Smirnov [BS03, Section 5] with a proof that a specific Lie algebra is the universal central extension in [BS03, Corollary 5.23].

**Lemma 6.2.1.** *If $G$ is a (Jordan-Kantor-like) sequence pair over $\Phi$, then the universal representation of $G$ is generated by the elements of the associated Jordan-Kantor pair.*

*Proof.* Suppose that $G_\sigma$ is in standard form. We prove that $(a, 0)_n = a_1^n/(n!)$. We shall denote $(a, 0)_n$ as $a_n$. We use equation (2.1) to get

$$a_i a_j = \binom{i+j}{i} a_{i+j},$$

for all $i$ and $j$. Using induction, one proves that $(n!)a_n = a_1^n$. So, we get $a_n = a_1^n/(n!)$. The same is true for elements $(0, b)_n$. Specifically, they can be rewritten using $(n!)(0, b)_{2n} = (0, b)_2$, and $(0, b)_{2n+1} = 0$. Hence, each $(a, b)_n$ is a polynomial in the $\sigma, 2\sigma$ graded elements of the associated Jordan-Kantor-like sequence pair. As the $(a, b)_n$ for $(a, b) \in G_\pm(\Phi)$ and $n \in \mathbb{N}$ generate the universal representation, we are finished. ∎

**Theorem 6.2.2.** *Let $P$ be a Jordan-Kantor pair over $\Phi$. The universal representation of the sequence pair associated to $P$ is isomorphic to $U(\tilde{L})$.*

*Proof.* We set $U = U(\tilde{L})$, namely the universal enveloping algebra of $\tilde{L}$. This is a $\mathbb{Z}$-graded co-commutative Hopf algebra, as it inherits the grading of $\tilde{L}$. For elements $(x_1, x_2) \in L_\sigma \oplus L_{2\sigma}$ (with $L$ the TKK Lie algebra of $P$), we consider the infinite dps $(1, x_1, x_1^2/2 + x_2, x_1^3/6 + x_1x_2/2, \ldots)$. Note that, as Proposition (6.1.6) and Remark (6.1.7) indicate, the algebra with those divided power series satisfies the conditions of Theorem (2.4.23). Hence, we conclude that there is a sequence pair representation in $U$.

Now, we use the corresponding morphism $\gamma$ from the universal sequence pair representation $V$ to $U$, to prove that $U$ is isomorphic to $V$. We know that there is a Lie algebra morphism $\tilde{L} \longrightarrow V$. Therefore, there exists a unique $\theta : U \longrightarrow V$. Note that $\theta \circ \gamma$ and $\gamma \circ \theta$ are the identity map on the generating elements of both algebras (namely the elements of the Jordan-Kantor pair). ∎

# 7 Hopf duals and algebraic groups

In this chapter, we apply and generalize the results of the second article of Faulkner [Fau04]. The only generalizing results are contained in the third and fifth section. This will link the theory developed by us to affine algebraic group schemes. Although the theory developed by Faulkner applies to broader contexts than only fields, we will assume, throughout this chapter, that $\Phi$ is a field.

## 7.1 Hopf duals

Suppose that $V$ is a $\Phi$-vector space. We want to endow $V$ with a topology. To achieve that, we consider a base $\mathcal{B}$ of linear subspaces of $V$ which we consider to be a base for the neighborhoods of 0. The fact that $\mathcal{B}$ forms a basis, means precisely that for each $k, l \in \mathcal{B}$ there exists $m \in \mathcal{B}$ such that $m \subset k \cap l$. This induces a **linear topological $\Phi$-vectorspace** structure $V_{\mathcal{T}}$, with $\mathcal{T}$ the topology generated by $\mathcal{B}$.

Between linear topological $\Phi$-vectorspaces, we only consider continuous maps. If we write $V$ we mean $V$ with the discrete topology, i.e. $\{0\}$ is the basis $\mathcal{B}$. So, we see that

$$\mathrm{Hom}(V_{\mathcal{T}}, W) = \{\phi \in \mathrm{Hom}(V, W) : \phi(I) = 0 \text{ for some } I \in \mathcal{B}\}.$$

We can identify this functor Hom, with a functor

$$\mathrm{Hom} : \mathbf{LTopVec}_{\Phi}^{op} \times \mathbf{LTopVec}_{\Phi} \longrightarrow \mathbf{Vec}_{\Phi},$$

where $\mathbf{LTopVec}_{\Phi}$ denotes the category of linear topological vectors spaces and $\mathbf{Vec}_{\Phi}$ the category of $\Phi$-vector spaces.

Now, we want to use these structures to dualize a Hopf algebra $H$ in a meaningful way. We set, for Hopf algebras $H$ with a linear base $\mathcal{B}$ generating a topology $\mathcal{T}$, $H^*_{\mathcal{T}} = \mathrm{Hom}(H_{\mathcal{T}}, \Phi)$. The operations on this new algebra, which we denote in the following equations with $A$, are given by

$$\mu_A = \mathrm{Hom}(\Delta_H, \mu_{\Phi}), \tag{7.1}$$
$$\eta_A = \mathrm{Hom}(\epsilon_H, \mathrm{Id}_{\Phi}), \tag{7.2}$$
$$\Delta_A = \mathrm{Hom}(\mu_H, \Delta_{\Phi}), \tag{7.3}$$
$$\epsilon_A = \mathrm{Hom}(\eta_H, \mathrm{Id}_{\Phi}), \tag{7.4}$$
$$S_A = \mathrm{Hom}(S_H, \mathrm{Id}_{\Phi}), \tag{7.5}$$

with $\Delta_{\Phi}(\lambda) = \mu_{\Phi}^{-1}(\lambda)$ for all $\lambda \in \Phi$. This gives $A$ a Hopf algebra structure by [Fau04, Theorem 3]. Suppose again that $H$ is a Hopf algebra. We want the operators $\epsilon, \Delta, S$ to be continuous for $H$, if we endow it with a linear topological vectorspace structure. We define, to identify a sufficient condition, an operator $\wedge$ on the linear subspaces, by setting

$$I \wedge J = \ker((\pi_I \otimes \pi_J) \circ \Delta),$$

with $\pi_I, \pi_j$ the projections $H \longrightarrow H/I$. The maps $\Delta, \epsilon, S$ are continuous given

$$\text{there is } K \in \mathcal{B} \text{ with } K \subset \ker(\epsilon), \tag{7.6}$$

$$\text{for } I, J \in \mathcal{B}, \text{ there is } K \in \mathcal{B} \text{ with } K \subset I \wedge J, \tag{7.7}$$

$$\text{for } I \in \mathcal{B}, \text{ there is } J \in \mathcal{B} \text{ with } J \subset S^{-1}(I). \tag{7.8}$$

A linear basis $\mathcal{B}$ of ideals satisfying the above conditions with each $H/I$ finite-dimensional, is called a **Hopf dualizing base**. This ensures the continuity of the operators $\Delta, \epsilon$ and $S$ on $H^*_{\mathcal{T}}$.

**Lemma 7.1.1 (Corollary 7, [Fau04]).** *If $\mathcal{B}$ is a family of algebra ideals of a Hopf algebra $H$ such that for $I, J \in \mathcal{B}$,*

1. *there is $K \in \mathcal{B}$ with $K \subset I \wedge J$,*
2. *there is $K \in \mathcal{B}$ with $K \subset S^{-1}(I)$,*
3. *$\epsilon(I) = 0$,*
4. *$H/I$ is finite-dimensional,*

*then $\mathcal{B}$ is a Hopf dualizing base.*

*Proof.* $\mathcal{B}$ is a linear base since $I \wedge J \subset I \cap J$. All other properties clearly hold. ∎

*Example* 7.1.2. Let

$$H = \bigoplus_{n=0}^{\infty} H_n$$

be a $\mathbb{Z}$-graded Hopf algebra so that each $H_n$ is finite-dimensional over $\Phi$. Let $I_m = \bigoplus_{n=m}^{\infty} H_n$, so

$$H/I_m \cong \bigoplus_{n=0}^{m-1} H_n.$$

These ideals satisfy the conditions of Lemma (7.1.1), so that $\mathcal{B} = \{I_m : m > 1\}$ forms a Hopf dualizing base. In this case, the continuous dual $H^*_{\mathcal{T}}$ is also the **graded dual** $H^g$. The graded dual of a graded vector space $\bigoplus_i V_i$ is $V^g = \bigoplus_i V_i^*$ with trivial action $V_i^*$ on $V_j$ for $i \neq j$. This means that if each of the $H_n$ is finite-dimensional, then $(H^g)^g = H$ as Hopf algebras.

Set

$$\bigwedge^{n} I = I \wedge \bigwedge^{n-1} I, \quad \bigwedge^{1} I = I.$$

**Theorem 7.1.3 (Theorem 8, [Fau04]).** *Let $\mathcal{F}$ be a family of algebra ideals in a Hopf algebra $H$ such that for all $F, K \in \mathcal{F}$, we have $J \wedge K \in \mathcal{F}$ and such that the image of*

$$(\pi_J \otimes \pi_K) \circ \Delta$$

*is a direct summand of $H/J \otimes H/K$. If $I \in \mathcal{F}$ is such that $\epsilon(I) = 0, S(I) \subset I$, and $H/I$ is finite-dimensional, then*

$$\mathcal{B}(I) = \left\{ \bigwedge^{n} I : n \geq 1 \right\}$$

*forms a Hopf dualizing base with topology $\mathcal{T}$. Moreover, $H^*_{\mathcal{T}}$ is generated as an algebra by*

$$\mathcal{Z}_{H^*}(I) = \{f \in H : f(I) = 0\}.$$

*Therefore, $H^*_{\mathcal{T}}$ is finitely generated as an algebra.*

## 7.2 Algebraic groups

**Definition 7.2.1.** A functor $F : \Phi\text{-}\mathbf{alg} \longrightarrow \mathcal{C}$ is a $\Phi$-**functor**. Such is functor is an **affine scheme** if there exists a commutative $\Phi$-algebra $\Phi[F]$ such that $F$ is equivalent to $\mathrm{Hom}_{\Phi\text{-}\mathbf{alg}}(\Phi[F], -)$. An affine scheme is **algebraic** if $\Phi[F]$ is a finitely generated algebra. A $\Phi$-functor, which is also an affine scheme, to **Grp** is a $\Phi$-**group scheme**. An **algebraic** $\Phi$-**group** is a $\Phi$-group scheme which is algebraic as an affine scheme.

If $G$ is a $\Phi$-group scheme, then $\Phi[G]$ is a commutative Hopf algebra and the product on $G(K)$ is given by

$$fg = \mu_K(f \otimes g) \circ \Delta,$$

with $\mu_K$ the multiplication on $K$. Conversely, any commutative Hopf algebra induces a group scheme in this way. So, $G_{H_\mathcal{T}} = \mathrm{Hom}_{\Phi\text{-}\mathbf{alg}}(H^*_\mathcal{T}, -)$ is a group scheme if $H$ is a cocommutative Hopf algebra. If $H, \mathcal{F}, I$ are as in Theorem (7.1.3) we denote the group scheme by $G_{H,I}$.

**Lemma 7.2.2 (Lemma 10, [Fau04]).** *If $I$ is an algebra ideal of finite codimension in a cocommutative Hopf algebra $H$ over $\Phi$ with $I \subset \ker(\epsilon)$ and $S(I) \subset I$, then $G_{H,I}$ is an algebraic $\Phi$-group. If $H'$ is a Hopf subalgebra of $H$ and $I' = I \cap H'$, then $G_{H',I'}$ is an algebraic $\Phi$-subgroup of $G_{H,I}$.*

Consider the Hopf algebra $\Phi[G]$ for a $\Phi$-group scheme $G$. We want to consider a suitable dual of $\Phi[G]$. Let $I = \ker(\epsilon)$ and assume that each $\Phi[G]/I^n, n > 0$ is a finite-dimensional vectorspace. This forms, as indicated in [Fau04, Example 6], a Hopf dualizing base. We set

$$\mathrm{Dist}(G) = \Phi[G]^*_\mathcal{T}.$$

We call this the **distribution or hyperalgebra** of $G$. The **Lie algebra**, as defined in [Jan87, Paragraph 7.7], of $G$ is

$$\mathrm{Lie}(G) = \{f \in \mathrm{Dist}(G) : f(1) = 0, f(I^2) = 0\},$$

this coincides with the set of primitive elements of Dist($G$). This is, as indicated in [DG70, II.§ 4.6.8] or [Mil13, Proposition 3.4], the usual Lie algebra associated to group schemes, namely the Lie algebra defined by the kernel of $G(\pi)$ with

$$\pi : \Phi[\epsilon] \longrightarrow \Phi : a + b\epsilon \longmapsto a,$$

with $\Phi[\epsilon]$ the dual numbers. We remark that each algebraic $\Phi$-group satisfies this condition. Indeed, $\Phi[G] = \Phi 1 \oplus \ker(\epsilon)$ is finitely generated as an algebra. So, we can choose generators $1, x_1, \dots, x_m$ with $x_i \in I$. Observe that 1 and the monomials in $x_i$ span $\Phi[G]$. We also note that $I$ is spanned by all monomials. So, $I^n$ is spanned by all monomials of length at least $n$. Hence, $\Phi[G]/I^n$ is finite-dimensional.

## 7.3 Finite-dimensional sequence $\Phi$-groups over fields

Let $G$ be a sequence $\Phi$-group. We identify it with a $\Phi$-group functor

$$K \longrightarrow G(K).$$

We note that this is, for finite-dimensional $G$, an algebraic $\Phi$-group. Namely, consider a basis $B$ for $G(\Phi)$. We see that if we take $A$ to be the ring $\Phi[B]$, i.e. the polynomial ring with variables in

$B$, that $G$ is equivalent to $K \mapsto \mathrm{Hom}_{\Phi\text{-}\mathbf{alg}}(A, K)$. Since $A$ is finitely generated, we see that this scheme is algebraic.

We can take the commutative Hopf algebra corresponding to the group structure. Consider the unique $f = G/H^1(\Phi) \otimes G/H^1(\Phi) \longrightarrow H^1(\Phi)$ corresponding to the bilinear form $\psi$ in $(a, b)(c, d) = (a + c, b + d + \psi(a, c))$. Using this $f$ we get that the coproduct

$$\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i + \tilde{f}^*(e_i),$$

where we used $f^*$ to denote the dual of $f$ going from $H^1(\Phi)^* \longrightarrow G/H^1(\Phi)^* \otimes G/H^1(\Phi)^*$ and $\tilde{f}^*$ to mean that we extended it with image 0 to the whole of $G$ (note that the tensor product interacts well with the duals since we are working with finite-dimensional vector spaces). The other maps $\epsilon, S$ are given by 0 and $-\mathrm{Id}$ on these generators. Note that $\Phi[G]$ has a natural grading on the generators. Namely, consider $e_i$ dual to $G/H^1$ to be 1-graded and $e_i$ as 2-graded if it is dual to $H^1$. So, we get a grading on $\Phi[G]$ if we set 1 to be 0-graded.

We want to identify the universal sequence $\Phi$-group representation with $\mathrm{Dist}(G)$. First, we construct the universal sequence group representation directly. Set $A$ to be the unital associative algebra generated by symbols $g_i$ for $g \in G(\Phi)$, $i \in \mathbb{N}$. We consider the quotient $A'$ with respect to the following relations, for $g, g' \in G(\Phi)$ and $\lambda \in \Phi$:

1. $g_0 = 1$,
2. $(\lambda \cdot g)_n = \lambda^n g_n$,
3. $(gg')_n = \sum_{i+j=n} g_i g'_j$,
4. $(1)_n = 0$, $n > 0$,
5. $h_i = 0$ for $h \in H^1$, $i$ odd,
6. $g_j g_i = \sum_{a+2b=i+j} \binom{a}{i-b} g_a (g_1^2 - 2g_2)_{2b}$,
7. $[g_j, g'_i] = \sum_{\substack{a+c=i \\ b+c=j \\ c \neq 0}} g'_a g_b [g, g']_{2c}$.

Note that this is a $\mathbb{Z}$-graded algebra. One easily endows $A'$ with a $\mathbb{Z}$-graded Hopf algebra structure by setting $\Delta(g_n) = \sum_{i+j=n} g_i \otimes g_j$, $\epsilon(g_i) = \delta_{i0}$, $S(g_n) = (g^{-1})_n$.

We generalize [Fau04, Lemma 12] from binomial divided power representations, to finite-dimensional sequence $\Phi$-groups over fields.

**Lemma 7.3.1.** *If $G$ is a finite-dimensional sequence $\Phi$-group over $\Phi$, then $\mathrm{Dist}(G) = (\Phi[G])^g \cong A'$ as $\mathbb{Z}$-graded Hopf algebras.*

*Proof.* We recall that the operations on $\mathrm{Dist}(G) = (\Phi[G])^g$ are given by equations (7.1), (7.2), (7.3), (7.4) and (7.5). We fix a basis of $G$, so that we can freely go to the dual vectorspace by mapping $a \mapsto a^*$. We assume that this basis corresponds to seeing $G$ as a direct sum of $G/H^1$ and $H^1$. We certainly have elements in $\mathrm{Dist}(G)$ which act on the 1-graded elements of $\Phi[G]$ by letting them evaluate a certain $g \in G(\Phi)$. We let $a = (a, 0) \in G(\Phi)$ act as

$$a(f) = f(a).$$

Also, if $h \in H^1(\Phi)$ the we let $h_n$ act on the $2n$-graded elements of $\Phi[G]$ by letting $f_1 \dots f_n$ evaluate to $f_1(h) \dots f_n(h)$ and mapping other $2n$-graded monomials in $\Phi[G]$ to 0. Specifically, the other $2n$-graded monomials are those which have more than $n$ contributing terms in the product, i.e. there are 1-graded generators which are part of the product. We note that the properties of [Fau04, Lemma 12] will hold for these elements $h_n$, as they are the elements of that lemma with action extended to $\Phi[G]$ by setting them to be 0 on monomials with contributing 1-graded generators. Moreover, they are contained in the center of $(\Phi[G])^g$, as a straightforward computation shows. We set $(a, \psi(a, a))_n = a_n$ and let $a_n$ act on $n$-graded monomials of $\Phi[G]$ by mapping

$$(f_1 \dots f_k)(f_1' \dots f_l') \text{ to } f_1(a) \dots f_k(a) f_1'(\psi(a, a)) \dots f_l'(\psi(a, a))$$

where we split the monomial in a product of 1-graded generators and a product of 2-graded generators.

As such, we define a sequence group representation of $G(\Phi)$ by setting

$$(a, b)_n = (a, \psi(a, a) + b')_n = \sum_{i+2j=n} a_i b_j'.$$

Firstly, this is clearly compatible with the scalar multiplication. We now prove that this is a group morphism. It is a group morphism on the elements of the form $(0, b)$. We could prove this directly, but we refer to [Fau04, Lemma 12]. Moreover, using this, the fact that all $(0, b)_i$ commute with everything and the definition of $(a, b)_n$ we get that $(a, b) \cdot (0, c) = (0, c) \cdot (a, b) = (a, b + c)$. So, we only need to see whether

$$(a, \psi(a, a)) \cdot (b, \psi(b, b)) \cdot (0, \psi(b, a)) = (a + b, \psi(a + b, a + b)),$$

holds for all $a, b$. So, we need to show that

$$\sum_{i+j+2k=n} a_i b_j \psi(b, a)_k = (a + b)_n$$

holds for all $n \in \mathbb{N}$. So, we let it act on a $n$-graded element $\prod_{i \in I_1} f_i \prod_{i \in I_2} f_i'$, where we split the product into a product of 1-generators generators and a product of 2-graded generators. We see that

$$\begin{aligned}
&\sum_{i+j+k=n} (a_i b_j \psi(b, a)_k)(\prod_{i \in I_1} f_i \prod_{i \in I_2} f_i') \\
&\quad = \sum_{i+j+k=n} \sum_{\substack{(p,q) \in P(I_1) \\ |p| \leq i \\ |q| \leq j}} (\prod_{f \in p} f(a) \prod_{f \in q} f(b))(a_{i-|p|} b_{j-|p|} \psi(b, a)_k)(\prod_{i \in I_2} f_i') \\
&\quad = \sum_{(p,q) \in P(I_1)} (\prod_{f \in p} f(a) \prod_{f \in q} f(b)) \sum_{i+j+k=n-|p|-|q|} (a_i b_j \psi(b, a)_k)(\prod_{i \in I_2} f_i') \\
&\quad = \sum_{(p,q) \in P(I_1)} (\prod_{f \in p} f(a) \prod_{f \in q} f(b))( \sum_{(k_{\psi(a,a)}, k_{\psi(a,b)}, k_{\psi(b,b)}, k_{\psi(b,a)}) \in P'(I_2)} \prod_{k_u} \prod_{f \in k_u} f(u)),
\end{aligned}$$

where $P(I)$ denotes the partitions of $I$ into 2 sets and $P'(I)$ the partitions of $I$ into 4 sets, now we justify the individual steps. The first step is using partial evaluation on the 1-graded generators using the definition of the multiplication. The second step is interchanging summations signs. The third step is then evaluating on the 2-graded generators also using the definition of the multiplication. The partition corresponds to a generator being fully evaluated by $a_i$, being partially evaluated in $a_i$ and partially evaluated in $b_j$, being fully evaluated in $b_j$ or being evaluated in $\psi(b, a)$. The possible partial evaluation comes from the fact that $\Delta(f_i') = f_i' \otimes 1 + 1 \otimes f_1' + \psi^*(f')$ and that $\psi^*(f')$ can be used to contribute in $a_i$ and in $b_j$. Evaluating using $(a + b)_n$ yields the same expressions. Specifically, it is evaluation in $(a + b)_n$ and then using linearity of the $f$'s and $\psi$ on $a + b$ (or $\psi(a + b, a + b)$) to get to the same expressions.

We also prove that this is a sequence $\Phi$-group representation. We must prove that the equations (2.1) and (2.2) hold. By [Fau04, lemma 12] we already know that if we see $H^1$, using Remark (2.2.4), as a sequence $\Phi$-group that the equations of the lemma hold. Moreover, we also know that the elements $h_n$ for $h \in H^1(\Phi)$ are contained in the center. So, we can assume that we are working with elements of the form $(a,0)_n$. In the following part of the proof we mean by $a_i = (a,0)_i$. For such elements, the proof of equation (2.1) is relatively easy. We compute that

$$a_j a_i = \sum_{c+2b=i+j} \binom{c}{i-b} a_c \psi(a,a)_b,$$

by seeing that they agree on the elements $(a^*)^{n-2i}(\psi(a,a)^*)^i$. This is sufficient, since the evaluation of other elements can be done by computing a projection onto these elements and evaluating, as all evaluations of generators will either be in $a$, in $\psi(a,a)$ or in 0. So, we can replace each generator a multiple of $a^*$ or $\psi(a,a)^*$ which evaluates the same on $a$ and $\psi(a,a)$. The binomial coefficient comes exactly from the fact that if you act on $(a^*)^c(\psi(a,a)^*)^b$, then there are $\binom{c}{i-b}$ ways how $(a^*)^c$ can split over $a_j, a_i$ to contribute, as $(\psi(a,a)^*)^b$ can only contribute by splitting up in two equal parts contributing in $a_j$ and in $a_i$.

We also prove equation (2.2). We remark that it is sufficient that the second equation holds for a generating set. So, we can assume that $a$ is orthogonal or equal to $b$ ($a^*(b) = 0$ or $a = b$). So, is the equation

$$a_i b_j = \sum_{\substack{k+m=j \\ l+m=i}} b_k a_l [a,b]_m$$

satisfied? If $a = b$, this is a consequence of the first equation. Specifically, the binomial coefficients $\binom{c}{i-b}$ and $\binom{c}{j-b}$ are equal in each term of the right hand side, as $(i-b)+(j-b) = c$. So, we can assume that $a^*(b) = b^*(a) = 0$. This makes it easy to work with a projection. Specifically, we map 1-graded generators $f$ using $f \longmapsto f(a)a^* + f(b)b^*$ and keep the 2-graded generators as they are. We note that the left hand side of the equation is determined by

$$(a^*)^{i-o}(b^*)^{j-o}(\psi(a,b)^*)^o \longmapsto 1$$

for all $o \leq \min(i,j)$, using the defined projection and a projection onto $\psi(a,b)^*$. For the right-hand side, we do not have a projection. However each product of 2-graded generators gets partitioned into a part which gets evaluated by $b_k a_l$ (i.e they are evaluated in $\psi(b,a)$) and a part which gets evaluated by $[a,b]_m$. We will show that this is the same as evaluating all 2-graded generators in $\psi(a,b)$. Specifically, set $o = \min(i,j)$. We look at what happens to the 2-graded generators contributing to an $i+j$-graded element. Specifically, we show that if there are at most $w \leq o$ contributing 2-generators, then we can evaluate using a projection on $(\psi(a,b)^*)^w$. If there are more, then we will show that the evaluation yields, necessarily 0.

There are at most $p = \min(k,l)$ evaluations in $\psi(b,a)$ and there are exactly $m$ evaluations in $[a,b]$ if we use a term in the right hand side to evaluate an $i+j$-graded element. Since $p+m = o = \min(i,j)$, we know that there are at most $o$ 2-graded generators in an element which does not evaluate to 0. Suppose that $w \leq o$. An $i+j$-graded element with $w$ 2-graded generators evaluates by considering all partitions $(\tilde{p},\tilde{m})$ of $w$ and evaluating the $p$ generators in $\tilde{p}$ in $\psi(b,a)$ and the $m$ generators of $\tilde{m}$ in $[a,b]$. As all $p+m = w$ are possible, this is the same as immediately evaluating all 2-graded generators in $\psi(a,b)$. So, we see that we evaluate all 2-graded generators in $\psi(a,b)$ and depending on the amount $w$ of 2-graded generators we evaluate $l-(w-m) = i-w$ times in $a$ and $k-(w-m) = j-w$ times in $b$ in a term of the right hand side. Hence, we have a sequence $\Phi$-group representation.

Now, we want to prove that this representation is a universal sequence $\Phi$-group representation. In what we have done, we identified a graded dual basis of $\Phi[G]$. Namely, order the chosen basis of $G$ to get $\{e_1, \ldots, e_n\}$ on the $G/H^1$ part. The ordering on the latter does not matter since these elements are all in the center. We denote the basis of $H^1$ by $h_1, \ldots h_m$. Then we consider the monomials of the form

$$(e_1, 0)_{n_1} \ldots (e_n, 0)_{n_{nn}} (h_1)_{m_1} \ldots (h_m)_{m_m}.$$

These, clearly, form a graded dual basis of $\Phi[G]^g$.

For each sequence $\Phi$-group representation $\rho : G \longrightarrow A$ there is a unique $\hat{\rho}$ going from these monomials to $A$. This extends to a unique linear map going from $\Phi[G]^g$ to $A$. We prove that it is an algebra morphism. This actually follows from the calculations we have done, to show that the representation is a sequence $\Phi$-group representation. Specifically, using the proved equations (namely equations (2.1) and (2.2)), there exists a unique way to write a product of two monomials in the basis as a sum of monomials in the basis. These equations hold in a general sequence $\Phi$-group representation. So, we know that $\hat{\rho}$ behaves like a morphism on a generating set. Therefore, it is a morphism.

We note, thus, that $\Phi[G]^g$ is the universal representation. We want to see that the Hopf algebra structure on $\Phi[G]^g$ coincides with the earlier constructed Hopf algebra structure on the universal representation. As in [Fau04, Lemma 12] $\Delta, \epsilon, S$ have all the right properties on $H^1$. So, the question remains whether they have all the right properties on the whole of $G$. We consider elements $(a, 0)_n$ and want to show that all the right properties are satisfied. We note that

$$a_i \otimes a_j (f_1 \otimes f_2) = a_{i+j}(f_1 f_2),$$

so that $\Delta(a_n) = \sum_{i+j=n} a_i \otimes a_j$, since

$$\Delta(a_n)(f_1 \otimes f_2) := \mu_\Phi^{-1}(a_n(f_1 f_2)).$$

Similarly, one sees that $\epsilon(a, b)_n = (a, b)_n(1) = 0$ for $n \neq 0$. Lastly, one notes that $S$ is already uniquely determined on the generators by the fact that $\mu \circ (\mathrm{Id} \otimes S) \circ \Delta = \eta \circ \epsilon$, so that $S(g_n) = (g^{-1})_n$ necessarily holds. ■

## 7.4 Group actions and comodules

Suppose that $G$ is an affine group scheme and $X$ is an affine scheme, then $G$ has a left action on $X$ if there exists a morphism $G \times X \longrightarrow X$ giving a group action of each $G(K)$ on each $X(K)$. This is equivalent with a left comodule structure:

$$\delta : \Phi[X] \longrightarrow \Phi[G] \otimes \Phi[X].$$

Specifically, one can go back and forth using

$$g \cdot x = \mu_K(g \otimes x) \circ \delta$$

and

$$\delta(v) = \mathrm{Id}_{\Phi[G]} \cdot (1 \otimes v).$$

Similarly one can define right actions and right comodules. If one considers a linear topological vector space $V_{\mathcal{T}}$, then we require that the action of $G$ is continuous.

The $\Phi$-**torus** $\Phi_m$ is the affine algebraic group scheme with $\Phi[\Phi_m] = \Phi[T, T^{-1}]$ and $\Delta(T) = T \otimes T, \epsilon(T) = 1, S(T) = T^{-1}$. As a consequence $\Phi_m(K)$ is the group of units of $K$. Since the torus is always a commutative group (as it has a cocommutative coordinate algebra), we do not bother in differentiating left and right group actions, we also do not need to differentiate between left and right comodules.

**Lemma 7.4.1 (Lemma 13, [Fau04]).** *A $\Phi$-module $V$ is a left $\Phi_m$ module if and only if $V$ is $\mathbb{Z}$-graded with*

$$g \cdot (v_i \otimes 1_k) = t^i (v_i \otimes 1_K)$$

*for $g \in \Phi_m(K)$ with $g(T) = t$ and $v_i \in V_i$. Moreover, $\Phi_m$-homomorphisms coincide with graded homomorphisms. A Hopf algebra $H$ is $\mathbb{Z}$-graded as a Hopf algebra if and only if $\Phi_m$ acts by automorphisms on $H$. Also, $\Phi_m$ acts as automorphisms of an affine group scheme $G$ if and only if $\Phi[G]$ is $\mathbb{Z}$-graded as a Hopf algebra.*

**Lemma 7.4.2.** *If $G$ is a finite-dimensional sequence $\Phi$-group, then the standard grading on the universal sequence group representation of $G$ corresponds to $G$ being a $\Phi_m$ module under the scalar multiplication of sequence groups.*

*Proof.* This is, mutatis mutandis, [Fau04, Lemma 14]. ■

**Lemma 7.4.3.** *If $H_{\mathcal{T}}$ is a finitely generated Hopf algebra, then $H$ is $\mathbb{Z}$-graded as a Hopf algebra, and if $\mathcal{T}$ is a linear base of graded subspaces, then $H^*_{\mathcal{T}}$ is a $\mathbb{Z}$-graded Hopf algebra.*

*Proof.* This is a weaker formulation of [Fau04, Corollary 19]. ■

## 7.5 Sequence pairs

Now that we have introduced all necessary concepts, we can generalize the last 2 theorems of the article in consideration [Fau04]. Recall that we identified, at the beginning of section 5.3 three subalgebras of the universal representation $U(G)$ of a sequence pair $G$, namely $\mathcal{X}, \mathcal{Y}, \mathcal{H}$.

**Theorem 7.5.1.** *If $G$ is a finite-dimensional Jordan-Kantor-like sequence pair over $\Phi$, $J$ is the kernel of the TKK representation, and*

$$I = \ker(\epsilon) \cap J \cap S(J),$$

*then $G' = G_{U(G),I}$ is an algebraic $\Phi$-group, with algebraic $\Phi$-subgroups*

$$U^+ = G_{\mathcal{X},I^+} \cong G_+, \quad U^- = G_{\mathcal{Y},I^-} \cong G_-, \quad H = G_{\mathcal{H},I^0},$$

*with $I^+ = \mathcal{X} \cap I$, etc.*

*Proof.* Since TKK$(G)$ is finite-dimensional, $J$ and $I$ have finite codimension. Moreover, since $U(G)$ is cocommutative, we get, by [MM65, Proposition 8.8], that $S^2 = \mathrm{Id}$. We also know that $\epsilon \circ S = \epsilon$. We see, furthermore, that $S(I) = I$. Lemma (7.2.2) shows that $G$ is an algebraic $\Phi$-group with algebraic subgroups $U^+, U^-$ and $H$. Note that $\Phi[G_+]^g$ is isomorphic to the subalgebra $\mathcal{X}$ of $U(G)$, since each representation of the sequence group $G_+$ can be extended to a representation of the sequence pair, by using the trivial representation for $G_-$. We note that $I_5 \subset I^+ \subset I_2$, with $I_m = \bigoplus_{n=m+1}^{\infty} (G_+)_n$, since the TKK-representation has $\rho_n^+ = 0$ for $n > 5$ and since $\ker(\rho_+^1) = 0$. As a consequence, using $I_n \wedge I_m = I_{n+m}$, the linear bases $\{\bigwedge^n I^+\}$ and $\{I_n\}$ determine the same topology on $\mathcal{X}$. By Lemma (7.3.1) we know that $U^+, U^-$ satisfy $\Phi[U^+] \cong (\Phi[G_+]^g)^g \cong \Phi[G_+]$ and $\Phi[U^-] \cong \Phi[G_-]$. We conclude that $U^\pm \cong G_\pm$. ■

*Remark* 7.5.2. Faulkner also proves that $H$ acts as automorphisms on the sequence pair $U^+, U^-$. However, we do not replicate the proof as it is quite long and it would require the introduction of a lot of additional concepts. With the theory developed in this thesis, it is possible to generalize what Faulkner proves to '$H$ acts as automorphisms if the characteristic of $\Phi$ is different from 2 and 3'.

We generalize the notion of an elementary action of the torus $\Phi_m$ on a separated $\Phi$-group sheaf $G$ as defined by Loos [Loo79]. However, we restrict ourselves to affine algebraic group schemes, as the result we are interested in only involves those. Suppose that $G$ is an affine algebraic group scheme with subgroup affine algebraic subgroup schemes $H, U^+, U^-$, and an action of $\Phi_m$ by automorphisms of $G$, such that

1. $H$ is fixed by $\Phi_m$,
2. $U^+, U^-$ are algebraic group schemes corresponding to sequence $\Phi$-group on which the action of $k_m$ corresponds to scalar multiplication (respectively, the inverse of scalar multiplication),
3. $\Omega = U^- H U^+$ is open in $G$,
4. $G$ is generated as a $\Phi$-group sheaf by $H, U^+, U^-$,

then we call this action on $G$ a **generalized elementary action** of $\Phi_m$.

For the next theorem, we first need to generalize [Loo79, Lemma 3.4].

**Lemma 7.5.3.** *For any generalized elementary action*

$$U^- \times H \times U^+ \longrightarrow G$$

*is an open embedding.*

*Proof.* It is sufficient to prove that this map is injective. We denote the embedding of $U^\pm$ in $G$ by exp. So, suppose that

$$\exp(y) h \exp(x) = h',$$

for some $y, x, h, h'$. We want to show that $x = 1 = y$ and $h = h'$. We let $t \in \Phi_m$ act upon the previous expression to get

$$\exp(t^{-1} \cdot y) h \exp(t \cdot x) = h'.$$

We suppose $t = 1 + \epsilon$ with $\Phi[\epsilon]$ the dual numbers. We see that

$$\exp((1 - \epsilon) \cdot y) h \exp((1 + \epsilon) \cdot x).$$

We compute that

$$(1 + \epsilon) \cdot x = x \times (\epsilon \cdot x) \times (\epsilon \cdot_{H^1} 2x_2 - x_1^2),$$

where we use $2x_2 - x_1^2$ to represent the element of $H^1$ corresponding to $x(-x)$. Similarly, we get

$$(1 - \epsilon) \cdot y = (-\epsilon \cdot_{H^1} 2y_2 - y_1^2) \times (-\epsilon \cdot y) \times y.$$

So, we conclude that

$$\exp(-\epsilon y_1, \epsilon(y_1^2 - 2y_2)) \exp(y) h \exp(x) \exp(\epsilon x_1, \epsilon(2x_2 - x_1^2)) = h',$$

where we introduced the first coordinates of $x$ and $y$ as $x_1$ and $y_1$. This implies that

$$\exp(-\epsilon y_1, \epsilon(y_1^2 - 2y_2))^{h'} = \exp(-\epsilon x_1, -\epsilon(2x_2 - x_1^2)).$$

However, this is impossible with $y_1 \neq 0 \neq x_1$, since

$$\begin{aligned} t \cdot \exp(-\epsilon x_1, -\epsilon(2x_2 - x_1^2)) &= t \exp(-\epsilon y_1, \epsilon(y_1^2 - 2y_2))^{h'} \\ &= t^{-1} \cdot \exp(-\epsilon y_1, \epsilon(y_1^2 - 2y_2))^{h'} \\ &= t^{-1} \cdot \exp(-\epsilon x_1, -\epsilon(2x_2 - x_1^2)) \end{aligned}$$

for all $t \in \Phi_m$, i.e. $tx_1 = t^{-1}x_1$ for all $t \in \Phi_m$. Using similar considerations using $\Phi[\epsilon']$ with $\epsilon'^4 = 0$, one shows that the second coordinates are also zero. ∎

**Theorem 7.5.4.** *If $G$ is an affine algebraic group scheme, then every generalized elementary action of $\Phi_m$ on $G$ gives a $\mathbb{Z}$-grading of Dist$(G)$ as a Hopf algebra, such that the induced $\mathbb{Z}$-grading of Lie$(G)$ is*

$$\mathit{Lie}(G) = \mathit{Lie}(U^-)_2 \oplus \mathit{Lie}(U^-)_1 \oplus \mathit{Lie}(H) \oplus \mathit{Lie}(U^+)_1 \oplus \mathit{Lie}(U^+)_2$$

*and there is a homogeneous divided power sequence over each $x \in$ Lie$(U^\pm)$. Moreover,*

$$(\mathit{Lie}(U^+), \mathit{Lie}(U^-))$$

*is a (Jordan-Kantor-like) sequence pair*[1].

*Proof.* By Lemma (7.4.1), the action of $\Phi_m$ on $G$ by automorphisms corresponds uniquely to a $\mathbb{Z}$-grading of $\Phi[G]$ as a Hopf algebra. Since $I = \ker(\epsilon)$ is a graded ideal, each $I^n$ is graded and Dist$(G)$ = $\Phi[G]^*_{\mathcal{T}}$ is graded by Lemma (7.4.3).

Lemma (7.5.3) shows that

$$U^- \times H \times U^+ \longrightarrow G$$

is an open embedding, so we get

$$\mathrm{Lie}(G) = \mathrm{Lie}(U^-) \oplus \mathrm{Lie}(H) \oplus \mathrm{Lie}(U^+).$$

Since $H$ is fixed by $\Phi_m$, we see that

$$\Phi[H] = \Phi[H]_0, \quad \mathrm{Dist}(H) = \mathrm{Dist}(H)_0, \quad \text{and} \quad \mathrm{Lie}(H) \subset \mathrm{Lie}(G)_0.$$

Since $\Phi_m$ acts on $U^+$ by the scalar multiplication of the sequence $\Phi$-group, the grading on $\Phi[U^+]$ is, by Lemma (7.4.2) the usual one. Thus

$$\mathrm{Lie}(U^+) \subset \mathrm{Dist}(U^+)_1 \oplus \mathrm{Dist}(U^+)_2,$$

and therefore Lie$(U^+) \subset$ Lie$(G)_1 \oplus$ Lie$(G)_2$. Similarly, one proves that Lie$(U^-) \subset$ Lie$(G)_{-1} \oplus$ Lie$(G)_{-2}$. Lemma (7.3.1) shows that there is a homogeneous divided power sequence over each element of Lie$(U^\pm)$. Finally, Theorem (2.4.23) proves that this is, in fact, a sequence pair. One easily shows that this representation of this sequence pair is a Jordan-Kantor-like sequence pair representation. ∎

[1] Technically we did not really define Jordan-Kantor-like sequence pairs if $1/2 \notin \Phi$. However, this should fall under any extended definition. Furthermore, later we will define them and this will fall under the extended definition.

# 8 Derivations of sequence pairs

## 8.1 Derivations

**Definition 8.1.1.** If $G$ is a sequence pair over $\Phi$ with a defining representation in $A$, then we mean by $G_K$, for $K \in \Phi$**-alg**, the sequence pair with defining representation in $A \otimes K$. Note that this is something different than $G(K)$, as the latter is not a sequence pair. It is, however, true that $G_K$ is fully determined from the representation of $G(K)$ in $A \otimes K$. We use the same notation for sequence groups $G'$.

Theorem (7.5.1) indicates that it is only natural to define the **derivations** as the pairs of graded $\Phi$-module endomorphisms $(\delta^+, \delta^-)$ of $G_+, G_-$ (we mean by this that $\delta^{\pm}(a,b) = (\delta_1^{\pm} a, \delta_2^{\pm} b)$) such that $(\mathrm{Id} + \epsilon\delta^\sigma)$ are sequence group automorphisms for $G_{\sigma\Phi[\epsilon]}$ with $\Phi[\epsilon]$ the dual numbers, and such that

$$(\mathrm{Id} + \epsilon\delta^\sigma) X_x(y) = X_{(\mathrm{Id}+\epsilon\delta^\sigma)x}((\mathrm{Id} + \epsilon\delta^{-\sigma})y)$$

holds strictly, for all operators $X = Q, T$ or even $P$ in the case of Jordan-Kantor-like sequence pairs. Formulated differently, we ask that $(\mathrm{Id} + \epsilon\delta^+, \mathrm{Id} + \epsilon\delta -)$ is an automorphism of the sequence pair $G_{\Phi[\epsilon]}$. The assumption that these $\delta$ must be graded, is there since we need compatibility with the scalar multiplication of the sequence groups.

A straightforward computation shows that $\delta(a,b) = (\delta_1 a, \delta_2 b)$ induces a sequence group morphism $\mathrm{Id} + \epsilon\delta$ if and only if $\delta_2\psi(a,b) = \psi(\delta_1 a, b) + \psi(a, \delta_1 b)$, with $\psi_\sigma$ the usual bilinear form associated with the product of the groups. Before we continue with $Q$, it is useful to rewrite the action of

$$\mathrm{Id} + \epsilon\delta,$$

namely, it maps an element $(a,b)$ to

$$(a + \epsilon\delta_1 a, b + \epsilon_1\delta_2 b) = (a,b)\cdot(\epsilon\delta_1 a, 0)\cdot(0, \epsilon(\delta_2 b - \psi(a, \delta_1 a))).$$

We will denote this composition as

$$(\mathrm{Id} + \epsilon\delta)x = x \cdot (\epsilon x') \cdot (\epsilon \cdot_H x'').$$

This is useful, since we cannot linearize expressions in sums of elements of $G$ well. However, we can linearize products of such elements. Using this composition, one computes that on $T$ the restriction becomes

$$\delta T_x y = T^{(2,1)}_{x,x'} y + T^{(1,2)}_{x,x''} y + T_x \delta y,$$

with $T^{(i,j)}$ the $(i,j)$-linearization of $T$. For the operator $Q$, it is not that easy. We can compute that

$$Q_x(a,b) = (Q^1_x a, Q^3_x a + Q^2_x b),$$

with $Q^1, Q^2$ as usual and $Q^3_x a$ varying by $Q^3_{tx}(sa) = t^4 s^2 Q^3_x a$ (we will see that this is a quadratic form in $a$). So, if we apply a sequence pair automorphism $(\mathrm{Id} + \epsilon\delta)$, we get

$$(Q^1_x a + \epsilon\delta_1 Q^1_x a, Q^3_x a + Q^2_x b + \epsilon\delta_2(Q^3_x a + Q^2_x b)) = Q_{(\mathrm{Id}+\epsilon\delta)x}((\mathrm{Id} + \epsilon\delta)(a,b)).$$

So, we compute the right hand side, in order to compare the coefficients of $\epsilon$. We get

$$\begin{aligned}&(Q_x^1(a+\epsilon\delta_1 a)+\epsilon Q_{x,x'}^{1,(1,1)}(a)+\epsilon Q_{x''}^1(a),\\ &Q_x^3(a+\epsilon\delta_1 a)+\epsilon Q_{x,x'}^{3,(3,1)}(a)+\epsilon Q_{x,x''}^{3(2,2)}a+Q_x^2(b+\epsilon\delta_2 b)+\epsilon Q_{x,x'}^{2,(3,1)}(b)+\epsilon Q_{x,x''}^{2,(2,2)}(b)),\end{aligned}$$

where the $(i,j)$ stands for $(i,j)$-linearization. Since $Q_x^3(a+\epsilon\delta_1 a)$ varies quadratically with the coefficient of $a$, we need to do some work to determine the term belonging to $\epsilon$. We denote this linearization, in the following restriction as $f_x(a,b)$. So, we have a sequence pair morphism if the following equations hold:

- $\delta_2\psi(a,b)=\psi(\delta_1 a,b)+\psi(a,\delta_1 b)$,
- $\delta T_x y=T_{x,x'}^{(2,1)}y+T_{x,x''}^{(1,2)}y+T_x\delta y$,
- $\delta_1 Q_x^1 a=Q_x^1\delta_1 a+Q_{x,x'}^{1,(1,1)}a+Q_{x''}^1 a$,
- $\delta_2 Q_x^3 a=f_x(a,\delta_1 a)+Q_{x,x'}^{3,(3,1)}a+Q_{x,x''}^{3,(2,2)}a$,
- $\delta_2 Q_x^2(h)=Q_x^2\delta_2 h+Q_{x,x'}^{2,(3,1)}h+Q_{x,x''}^{2,(2,2)}h$,

for all $x\in G_\sigma(K)$, $y=(a,h)\in G_{-\sigma}(K)$, $b\in G/H_{-\sigma}^1(K)$.

We linearize $Q_x^3(a)$ to $a$ to determine what $f_x(a,\delta_1 a)$ should be. Specifically, we compute what the term belonging to $t$ is in $Q_x^3(a+tb)$. To achieve that, we compute

$$Q_x(y\cdot z)Q_x(y)^{-1}Q_x(z)^{-1}.$$

we note that the first coordinate is necessarily 0, since $Q_x^1$ is linear. We recall from Lemma (2.1.9) that $\mathrm{ad}_x^{(n)}(ab)=\sum_{i+j}\mathrm{ad}_x^{(i)}(a)\mathrm{ad}_x^{(j)}(b)$. We want to work with elements $[a,b]$ and $(a,b)$ in a defining representation. So, we denote these elements for $x,y$ as $(a,b)$ for $x,z$ as $(a,b)'$ and for $x,y\cdot z$ as $(a,b)''$, we do the same for the elements $[a,b]$. So, we compute

$$(4,2)''=\mathrm{ad}_x^{(4)}(y\cdot z)_2=\mathrm{ad}_x^{(4)}(y_2+z_2+y_1z_1)=(4,2)+(4,2)'+[1,1][3,1]'+[2,1][2,1]'+[3,1][1,1]'.$$

Therefore, using the fact that elements of the form $[a,1]$ are linear functions in the first coordinate of the second dependency, we get

$$\begin{aligned}[4,2]''&=(4,2)''-[1,1]''[3,1]''\\ &=(4,2)+(4,2)'+[[3,1],[1,1]']-[1,1]'[3,1]'-[1,1][3,1]+[2,1][2,1]'\\ &=[4,2]+[4,2]'+[[3,1],[1,1]']+[2,1][2,1]'\\ &=(Q_x(y)Q_x(z))_2+[[3,1],[1,1]']\end{aligned}$$

So, we see that

$$Q_x(y\cdot z)=Q_x(y)Q_x(z)(0,-V_{x,z}T_xy).$$

We conclude that

$$f_x(a,\delta_1 a)=-Q_x^2\psi(a,\delta_1 a)+\psi(Q_x^1 a,Q_x^1\delta_1 a)-V_{x,\delta_1 a}T_xa,$$

as

$$(Q_x^1a,Q_x^3a+Q_x^2b)(Q_x^1c,Q_x^3c+Q_x^2d)=(Q_x^1(a+c),Q_x^3(a+c)-f_x(a,c)+Q_x^2(b+d)+\psi(Q_x^1a,Q_x^1c))$$

should equal

$$(Q_x^1(a+c), Q_x^3(a+c) + Q_x^2(b+d+\psi(a,c)) + V_{x,c}T_x a).$$

If $1/2 \in \Phi$ we can assume that we are working with sequence groups in standard form, so that $\psi(a, \delta a) = [a, \delta a]/2$. We know how $Q_x^2$ interacts with the group commutator. Namely, we can use that

$$\begin{aligned}(Q_x^2[a,b])_2 &= \mathrm{ad}_x^{(4)}[a,b]_2 \\ &= [(T_x a)_2, [x_1, b_1]] + [(Q_x a)_1, (Q_x b)_1] + [[x_1, a_1], (T_x b)_2] \\ &= (-V_{x,b}T_x a)_2 + (V_{x,a}T_x b)_2 + [Q_x a, Q_x b]_2.\end{aligned}$$

This lets us rewrite

$$f_x(a, \delta_1 a) = -1/2(V_{x,\delta_1 a}T_x a + V_{x,a}T_x \delta_1 a).$$

Note that $f$ is a symmetric bilinear form. It is possible to prove that $f_x$ is bilinear if $1/2 \notin \Phi$ as well, making use of the fact that $\psi(a,b) - \psi(b,a) = [a,b]$ to compute $f_x(a,b) - f_x(b,a)$.

*Remark* 8.1.2. Note that it is not at all obvious that the operators $V_{x,y}$, for any Jordan-Kantor-like sequence pair, satisfy the previous equations. However, if $1/6 \in \Phi$ it is relatively easy to prove without any computation. Specifically, consider $B(x, \epsilon y) = 1 + \epsilon[1,1]$ for $x \in G_{-\sigma}(\Phi)$ and $y \in G_\sigma(\Phi)$. This acts, using conjugation in the TKK representation, as an automorphism. Hence, ad $[1,1] = V_{x,y}$ is a derivation. Similarly, one can show that $[2,2]$ induces a derivation if $y \in H_\sigma^1(\Phi)$.

Note that the derivations are closed under linear combinations. Suppose that $\delta, \delta'$ are derivations. We look at $\Phi[\epsilon, \epsilon']$ with $\epsilon^2 = \epsilon'^2 = 0$. We know that $(\mathrm{Id} + \epsilon\delta), (\mathrm{Id} + \epsilon'\delta')$ are automorphisms of $G_{\Phi[\epsilon,\epsilon']}$. Therefore,

$$(\mathrm{Id} + \epsilon\epsilon'[\delta, \delta']) = [(\mathrm{Id} + \epsilon\delta), (\mathrm{Id} + \epsilon'\delta')]$$

is an automorphism of $G(\Phi[\epsilon, \epsilon'])$. We note that this automorphism maps the subgroup

$$G(\Phi[\epsilon \cdot \epsilon']) \cong G(\Phi[\epsilon])$$

to itself. Moreover, the action on $G(\Phi[\epsilon])$ is exactly given by $\mathrm{Id} + \epsilon[\delta, \delta']$. This proves that $[\delta, \delta']$ is a derivation of $G$. Hence, we know that the derivations of a sequence group form a Lie algebra over $\Phi$. We call this algebra **the derivation algebra** of $G$. We follow Loos [Loo75] in this naming convention for Jordan pairs, and do not make a distinction between the structure algebra and the derivation algebra if there is no clear unit.

## 8.2 TKK Lie algebras and representations

Suppose that we have a sequence pair $G$ with an additional operator $P$ satisfying Definition (4.3.1), i.e. we have a Jordan-Kantor-like sequence pair without the assumption that $1/2 \in \Phi$. We will try, by making use of the derivation algebras, to determine exactly what the Jordan-Kantor-like sequence pairs should be. We set

$$\overline{\mathbf{InDer}(G)} = \langle [1,1] | x \in G_+(\Phi), y \in G_-(\Phi) \rangle + \langle [2,2] | x \in G_\sigma(\Phi), h \in H_{-\sigma}^1(\Phi) \rangle,$$

i.e. the linear combinations of the mentioned elements. We will later see that this is a Lie algebra under certain conditions.

**Definition 8.2.1.** A **Jordan-Kantor-like sequence pair** (with $1/2$ not necessarily in $\Phi$) is a sequence pair $G$ with additional operator $P$ with a defining representation, satisfying additional restrictions (4.4) and (4.5) (The restrictions for a Jordan-Kantor-like sequence pair if $1/2 \in \Phi$), such that the conjugation with $B(sx, ty) = 1 + st[1,1] + s^2t^2[2,2] + \ldots$ induces an automorphism of the sequence pair for all $x \in G_\pm(\Phi), y \in G_\mp(\Phi)$. A **Jordan-Kantor-like sequence pair representation** is a sequence pair representation satisfying these additional restrictions and if it enjoys the same conjugation action as the defining representation (i.e. conjugation with $B(sx, ty)$ is an integral part of the structure).

*Remark* 8.2.2. This definition coincides with the previous definition of a Jordan-Kantor-like sequence pair with $1/6 \in \Phi$. It might be true that the new definition is a bit more restrictive in the case that $1/3 \notin \Phi$. If this is the case, then we want the new definition.

**Theorem 8.2.3.** *The universal Jordan-Kantor-like sequence pair representation of a Jordan-Kantor-like sequence pair $G$ is a $\mathbb{Z}$-graded Hopf algebra.*

*Proof.* We first consider the universal representation $U$ of the Jordan-Kantor-like sequence pair $G$ without the conjugation action (This is the universal sequence pair representation with the additional Jordan-Kantor-like sequence pair restrictions divided out). Lemmas (4.1.1) , (4.1.3) and (4.1.5) prove that the Jordan-Kantor-like sequence pair representations of $G$ without the conjugation action definitely form a sensible collection of representations. Notice that these lemmas also imply that $B(sx, ty)$ can be interpreted as an operator which is an integral part of the representations (i.e. it is compatible with algebra morphisms, representations $\rho \otimes \xi$ and representations $\rho \circ (\cdot^{-1})$ and even with the adjoint representation). We can apply Theorem (4.1.12) to prove that $U$ is a $\mathbb{Z}$-graded cocommutative Hopf algebra.

Now we consider the conjugation action. The question is, whether this action is preserved. We already know that $B = B(sx, ty)$ is a well-determined element of $U[[s,t]]$. In fact, it is sufficient to ensure that the conjugation action coincides with the conjugation action in the defining representation. So, we know that the divided power series $s_1 = \exp(z)^B$ should be the divided power series $s_2 = \exp(z^B)$. This is equivalent to requiring that $s_1 \times S(s_2) = (1)$, as sequences in the sequence group formed by all divided power series. We note that $s_3 = s_1 \times S(s_2)$ is a well defined divided power series. So, we need to ensure that $(s_3)_n = 0$ for $n \geq 1$. This is, clearly, necessary and sufficient. We prove that the ideal $I$ generated by these $(s_3)_n$ is a Hopf ideal. Firstly, it is clearly a coideal. Secondly $\epsilon(I) = 0$ as this is the case for all generators. Thirdly, $S(I) \subset I$ since the inverse of $s_3$, namely $S(s_3)$, can be computed using the usual algorithm for computing the inverse of a power series. This algorithm ensures that $S(s_3)_n$ is a polynomial in the $(s_3)_i$ and if $n > 0$ we know that in each contributing term there is at least one $(s_3)_i$ with $i > 0$. Note that this Hopf ideal is compatible with the grading.

Now we define $J$ as the ideal formed by all these ideals $I$ for all $x$ and $y$. Note that $J$ is a Hopf ideal of $U[[s,t]]$ instead of $U$ and that $J$ is generated by (possibly infinite sums) of homogeneous polynomials $s^at^bp$ with $p \in U$. Take the submodules $K_{a,b} = \{u \in U | s^at^bu \in J\}$ and note that $\cup_{a,b} K_{a,b}$ forms a Hopf ideal as well. We note that $U/K$ is the universal representation as $K$ is the minimal ideal of $U$ which ensures that $J$ is trivial in $U/K[[s,t]]$ and since each representation (with conjugation) $\rho : G \longrightarrow A$ induces $\xi : U \longrightarrow A$ which, if extended, factors through $U[[s,t]]/J \longrightarrow A[[s,t]]$ so if $\xi(k) \neq 0$ in $A$ for $k \in K_{a,b}$, then $0 = \xi(ks^at^b) = \xi(k)s^at^b \neq 0$, which is a contradiction. ■

*Remark* 8.2.4. We note that we could have proved that the representations with conjugation action form a sensible collection of representations. However, this would not help much, since we should still identify the (Hopf) ideal $K$ to construct the universal representation.

We set $\mathbf{InDer}(G) = \overline{\mathbf{InDer}}(G)/\sim$ with the equivalence determined by $\delta = \delta'$ if $1 + \epsilon\delta = 1 + \epsilon\delta'$ as automorphism of the pair. Note that we now know that this forms a Lie algebra, as the inner derivations correspond to automorphisms $B(x, \epsilon \cdot y) = 1 + \epsilon[i, i]$ (with $\cdot$ either the group scalar multiplication or the module scalar multiplication and $i = 1, 2$, depending on whether $y$ is a part of $H^1$ or not) and since this conjugation action is part of the structure of the Jordan-Kantor-like sequence groups we know that it is, not only, a subalgebra, but in fact an ideal of the derivation algebra. For general derivation algebras $\mathcal{D}$ containing the inner derivations, we can now consider a TKK representation in the endomorphism algebra of the 5-graded Lie algebra

$$\mathrm{TKK}(G, \mathcal{D}) = (H^1_-)_2 \oplus (G_-)_1 \oplus \mathcal{D} \oplus (G_+)_1 \oplus (H^1_+)_2.$$

The brackets involving elements $d$ of $\mathcal{D}$ are determined by $l^{1+\epsilon d} = l + \epsilon[l, d]$. Note that this Lie algebra is defined from $\tilde{L} = (H^1_-)_2 \oplus (G_-)_1 \oplus \overline{\mathbf{InDer}(G)} \oplus (G_+)_1 \oplus (H^1_+)_2$ contained in the universal representation (upon which we have the usual action). Then replacing $\overline{\mathbf{InDer}(G)}$ with $\mathbf{InDer}(G)$, which we will see to be compatible with the action, and then adding the elements of $\mathcal{D}$. So, we still need to define the representation in the endomorphism algebra of this Lie algebra. We only need to think about actions upon elements of $\mathcal{D}$. We consider $D \in \mathcal{D}$ and $x \in G_\sigma(\Phi)$. We see that

$$[1 + \epsilon D, \exp(x)^{-1}] = (1 - \epsilon D)\exp(x)(1 + \epsilon D)\exp(x^{-1}),$$

acts as

$$\exp(x)^{1+\epsilon D}\exp(x)^{-1} = \exp(\epsilon \cdot_{Mod} D'x),$$

with $D'x = Dx - \psi(D_1 x_1, x_1)$. Hence, we get that

$$1 + \epsilon D^{\exp(x)^{-1}} = \exp(x)(1 + \epsilon D)\exp(x)^{-1} = [1 + \epsilon D, \exp(x)^{-1}] + \epsilon D = \exp(\epsilon \cdot_{Mod} D'x) + \epsilon D.$$

So, we get as action

$$D^{\exp(x)^{-1}} = D + D'x.$$

We know that $x_n \cdot D$ must be the $n$-graded component of $D^{\exp(x)^{-1}}$. Hence we have a morphism from the universal representation to the endomorphism algebra of TKK($G, \mathcal{D}$). To see that it is well defined, first consider the central extension of TKK($G, \mathbf{InDer}(G)$) contained in the universal representation of $G$. Thereupon, we have a well-defined action using the adjoint representation. This induces an action on $L$ = TKK($G, \mathbf{InDer}(G)$) (the action on $\mathbf{InDer}(G)$ coincides with the computed action for a general $\mathcal{D}$). Now, we can use this to define the action on TKK($G, \mathcal{D}$). Specifically, each $D \in \mathcal{D}$ is an endomorphism of $L$, and we identified the action of $\exp(x)$ on $D$ in the endomorphism algebra and realised that $\exp(x) \cdot D - D \in L$ (i.e. there exists a unique sensible element $x'$ so that $\exp(x) \cdot D - D = \operatorname{ad} x'$, namely $x' = D'x$). So, we can add $D$ to $L$ without any problem. We can do this for all the $D \in \mathcal{D}$ at the same time to get TKK($G, \mathcal{D}$). So, we proved:

**Theorem 8.2.5.** *Let $G$ be a Jordan-Kantor-like sequence pair. For each derivation algebra $\mathcal{D}$ of $G$ containing the inner derivations, $L$ = TKK($G, \mathcal{D}$) is a 5-graded Lie algebra and $G$ has a Jordan-Kantor-like sequence pair representation in the endomorphism algebra of $L$.*

# A Nederlandstalige samenvatting

In deze thesis ontwikkelen we enkele concepten, namelijk 'sequence $\Phi$-groups' en 'sequence pairs', die ons toelaten om enkele resultaten van Faulkner [Fau00] en [Fau04] te veralgemenen van Jordan paren naar Jordan-Kantor paren. Gebruik makend van deze concepten slagen we erin om correspondenties te leggen tussen (1) Hopf algebras, (2) Jordan-Kantor paren, (3) Lie algebras, (4) algebraïsche groepen. We zullen deze linken hier uitleggen aan de hand van Jordan-Kantor-achtige sequence pairs. We zullen verwijzen naar enkele resultaten, maar soms zullen deze gaan over sequence pairs in plaats van Jordan-Kantor-achtige sequence pairs. Dergelijke resultaten kunnen altijd eenvoudig uitgebreid worden tot Jordan-Kantor-achtige sequence pairs. In wat volgt duiden we met $\Phi$ de commutatieve ring met eenheid aan (soms zullen we enkele extra voorwaarden formuleren) waarover we werken.

Ten eerste bewijzen we dat elke cocommutatieve $\mathbb{Z}$-gegradeerde Hopf algebra $H$, waarvan de primitieve elementen $5$-gegradeerd zijn en zodat er bovendien voldoende divided power series zijn (Definitie (1.5.5)), een sequence pair induceert (Stelling (2.4.23)). Omgekeerd, als we werken over een veld $\Phi$ met karakteristiek verschillend van $2$ en $3$, dan weten we dat de universele representatie van een Jordan-Kantor-achtig sequence pair een dergelijke Hopf algebra is (Gevolg (5.3.15)). Bovendien kunnen we garanderen, ongeacht of $\Phi$ een veld is, dat de universele representatie een $\mathbb{Z}$-gegradeerde cocommutateve Hopf algebra is (Stelling (8.2.3)).

Van een Jordan-Kantor-achtig sequence pair kunnen we de TKK Lie algebras $L$ en de sequence pair representaties in de endomorfismen algebra van $L$ beschouwen (Stelling (8.2.5)) voor de versie zonder assumpties op de invertibiliteit van de scalairen). Deze Lie algebra's zijn altijd $5$-gegradeerd. Omgekeerd kunnen we met een $5$-gegradeerde Lie algebra $L$ altijd een Jordan-Kantor paar $P$ associëren. Als $1/6 \in \Phi$, dan weten we dat het afbeelden van een Jordan-Kantor-achtig sequence pair op het overeenkomstig Jordan-Kantor paar een injectieve afbeelding is. Als $1/30 \in \Phi$, dan is het een bijectie (Gevolg (4.3.4)). Omgekeerd, als $1/30 \in \Phi$ kunnen we met elk Jordan-Kantor paar een Jordan-Kantor-achtig sequence pair associëren. Als $1/5 \notin \Phi$ is het enigszins subtieler. Eenvoudigst geformuleerd is de (nodige en voldoende) voorwaarde opdat er een overeenkomstig Jordan-Kantor-achtig sequence pair is, dat alle $\exp(x)$ automorfismen zijn in plaats van slechts endomorfismen (Stelling (2.4.8)).

Ook leggen we de link met algebraïsche groepen. Het is een veralgemening van de link gemaakt door Loos [Loo79], alhoewel onze veralgemening nog enigszins verbreed moet worden om werkelijk een volwaardige veralgemening te zijn. Specifiek introduceren we de notie van een veralgemeende elementaire actie. Hiermee kunnen we, indien we werken over een veld $\Phi$ en de sequence groups eindig dimensionaal zijn, heen en weer gaan tussen bepaalde algebraïsche groepen en Jordan-Kantor-achtige sequence pairs (Sectie 7.5).

Naast het vele heen en weer tussen verschillende algebraïsche structuren hebben we ook enkele relatief eenvoudige en tastbare voorbeelden gegeven van sequence pairs over ringen $\Phi$ die niet noodzakelijk $1/6$ bevatten. Hiervoor hebben we speciale sequence pairs (Hoofdstuk 3) onderzocht. We hebben gezien dat speciale sequence pairs heel eenvoudig toelaten, als $1/3 \notin \Phi$ zit (maar $1/2$ wel in $\Phi$), om bepaalde sequence pairs te construeren. Bijgevolg induceren, over dergelijke $\Phi$, alle associatieve algebra's met involutie een sequence pair (als uitbreiding van hoe ze dat doen als structureerbare algebra's). Bovendien hebben we indien $1/2 \notin \Phi$ bewezen dat bepaalde families van associatieve algebra's sequence pairs induceren. Hieronder vallen de separabele velduitbreidingen van graad 2 met Galois involutie en de quaternionenalgebra's.

Ook zijn we er ook in geslaagd (Sectie 4.4) om uit structureerbare algebra's gevormd uit een associatieve algebra $A$ en een rechts $A$-moduul $M$ met een hermitische vorm $M \times M \longrightarrow A$, indien $1/3 \notin \Phi$, ook sequence pairs te induceren. Net zoals bij de speciale sequence pairs wordt het indien $1/2$ niet voorhanden is een stuk subtieler.

Ten slotte zijn we erin geslaagd om indien $\Phi$ een veld is van karakteristiek $0$, een andere beschrijving te geven van de universele representatie van een Jordan-Kantor-achtig sequence pair $P$. Specifiek, de universele representatie is isomorf aan de universele enveloping algebra van de universele centrale cover van de TKK Lie algebra $L$ gerelateerd aan $P$ (Stelling (6.2.2)).

# B Homogeneous maps

We introduce some concepts and a theorem from [Fau00, Appendix A] pertaining to homogeneous maps.

Let $V$ and $W$ be modules over a unital, commutative associative ring $\Phi$. If $f : V \longrightarrow W$ is constant, we call $f$ **Homogeneous of degree** $0$. For each $n \geq 1$ we shall recursively define $f : V \longrightarrow W$ to be **homogeneous of degree** $n$ with $(i, j)$-**linearization** $f_{ij} : V \times V \longrightarrow W$, with $i + j = n, i, j \geq 1$, provided that for all $\lambda \in \Phi$, $u, v, w \in V$,

1. $f(\lambda v) = \lambda^n f(v)$,
2. $f(u + v) = f(u) + f(v) + \sum_{i+j=ni,j\geq 1} f_{ij}(v, v)$,
3. $u \mapsto f_{lk}(u, w)$ is homogeneous of degree $l$ with $(i, j)$-linearization $(u, v) \mapsto f_{ijk}(u, v, w)$,
4. $f_{ij}(v, v) = \binom{n}{i} f(v)$ for $i + j = n$, $i, j \geq 1$,
5. $f_{ij}(u, v) = f_{ji}(v, u)$ for $i + j = n$, $i, j \geq 1$,
6. $f_{ijk}(u, v, w) = f_{ikj}(u, w, v)$ for $i + j + k = n$, $i, j, k \geq 1$.

*Remark* B.1.1. Note that this is a generalization of the definition of a quadratic map. Namely, $f$ is homogeneous of degree 2 if and only if $f(\lambda v) = \lambda^2 v$, and

$$f_{11}(u, v) = f(u + v) - f(u) - f(v),$$

is bilinear.

We can further define linearizations for homogeneous maps $f$ of degree $n$. Specifically, we can define $f_{i_1,i_2,i_3\cdots i_k}$ as the $(i_1, i_2)$ linearization of $f_{i_1+i_2,i_3\cdots i_k}$.

**Theorem B.1.2.** *Let $f : V \longrightarrow W$ be homogeneous of degree $n$, for $\Phi$-modules $V$ and $W$, with linearizations $f_{i_1\cdots i_k}$ and let $\Omega$ be an extension ring of $\Phi$. If $(\tilde{V}, \tilde{W})$ is either $(V \otimes \Omega, W \otimes \Omega)$ or $(V[[s]], W[[s]])$ then there is a unique homogeneous map of degree $n$, $\tilde{f} : \tilde{V} \longrightarrow \tilde{W}$ with linearizations $\tilde{f}_{i_1\cdots i_k}$ which extend $f$ and $f_{i_1\cdots i_k}$.*

*Proof.* This is [Fau00, Theorem 35]. ∎

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